# Math Majors of America Tournament for High Schools

## 2015 Mixer Test

Team : ________

1. ________

2. ________

3. ________

4. ________

5. ________

6. write on back

7. write on back

8. ________

9. ________

10. write on back

11. ________

– 75 minutes
– no calculators
– simplify answers

1. Let $a_0, a_1, \ldots a_n$ be such that $a_n \neq 0$ and

$$(1+x+x^3)^{341}(1+2x+x^2+2x^3+2x^4+x^6)^{342} = \sum_{i=0}^{n} a_i x^i.$$

Find the number of odd numbers in the sequence $a_0, a_1, \ldots a_n$.

2. Let $F_0 = 1, F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$. Let $P(x) = \sum_{k=0}^{99} x^{F_k}$. The remainder when $P(x)$ is divided by $x^3 - 1$ can be expressed as $ax^2 + bx + c$. Find $2a + b$.

3. Let $a_n$ be the number of permutations of the numbers $S = \{1, 2, \ldots, n\}$ such that for all $k$ with $1 \leq k \leq n$, the sum of $k$ and the number in the $k$th position of the permutation is a power of 2. Compute $a_{2^0} + a_{2^1} + \cdots + a_{2^{20}}$.

4. Three identical balls are painted white and black, so that half of each sphere is a white hemisphere, and the other half is a black one. The three balls are placed on a plane surface, each with a random orientation, so that each ball has a point of contact with the other two. What is the probability that at at least one point of contact between two of the balls, both balls are the same color?

5. Compute the greatest positive integer $n$ such that there exists an odd integer $a$, for which $\dfrac{a^{2^n} - 1}{4^{4^4}}$ is not an integer.

6. You are blind and cannot feel the difference between a coin that is heads up or tails up. There are 100 coins in front of you and are told that exactly 10 of them are heads up. On the back of this paper, explain how you can split the otherwise indistinguishable coins into two groups so that both groups have the same number of heads.

7. On the back of this page, write the best math pun you can think of. You'll get a point if we chuckle.

8. Pick an integer between 1 and 10. If you pick $k$, and $n$ total teams pick $k$, then you'll receive $\frac{k}{10n}$ points.

9. There are four prisoners in a dungeon. Tomorrow, they will be separated into a group of three in one room, and the other in a room by himself. Each will be given a hat to wear that is either black or white – two will be given white and two black. None of them will be able to communicate with each other and none will see his or her own hat color. The group of three is lined up, so that the one in the back can see the other two, the second can see the first, but the first cannot see the others. If anyone is certain of their hat color, then they immediately shout that they know it to the rest of the group. If they can secretly prove it to the guard, they are saved. They only say something if they're sure. Which person is sure to survive?

10. Down the road, there are 10 prisoners in a dungeon. Tomorrow they will be lined up in a single room and each given a black or white hat – this time they don't know how many of each. The person in the back can see everyone's hat besides his own, and similarly everyone else can only see the hats of the people in front of them. The person in the back will shout out a guess for his hat color and will be saved if and only if he is right. Then the person in front of him will have to guess, and this will continue until everyone has the opportunity to be saved. Each person can only say his or her guess of "white" or "black" when their turn comes, and no other signals may be made. If they have the night before receiving the hats to try to devise some sort of code, how many people at a minimum can be saved with the most optimal code? Describe the code on the back of this paper for full points.

11. A few of the problems on this mixer contest were taken from last year's event. One of them had fewer than 5 correct answers, and most of the answers given were the same incorrect answer. Half a point will be given if you can guess the number of the problem on this test that corresponds to last year's question, and another .5 points will be given if you can guess the very common incorrect answer.