Math Majors of America Tournament for High Schools
2015 Mixer Test Solutions

1. Let $a_0, a_1, \dots a_n$ be such that $a_n \neq 0$ and

$$(1+x+x^3)^{341}(1+2x+x^2+2x^3+2x^4+x^6)^{342} = \sum_{i=0}^{n} a_i x^i.$$

Find the number of odd numbers in the sequence $a_0, a_1, \dots a_n$.

Answer: 3

Solution: The expression reduces to $(1+x+x^3)^{2^{10}}$. The number of odd coefficients doesn't change each time you square this, so there's still 3 after squaring 10 times.

2. Let $F_0 = 1, F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$. Let $P(x) = \sum_{k=0}^{99} x^{F_k}$. The remainder when $P(x)$ is divided by $x^3 - 1$ can be expressed as $ax^2 + bx + c$. Find $2a + b$.

Answer: 112

Solution: Counting the Fibonacci sequence mod 3 gives a period of 8: 1, 1, 2, 0, 2, 2, 1, 0 and repeats. We know a counts the number of terms with powers $2 \pmod 3$ in the first 100 terms, b counts the number of terms 1 (mod 3), and c counts the number of terms 0 (mod 3). We can compute using that periodic sequence that $(a, b, c) = (37, 38, 25)$.

3. Let a_n be the number of permutations of the numbers $S = \{1, 2, \dots, n\}$ such that for all k with $1 \leq k \leq n$, the sum of k and the number in the kth position of the permutation is a power of 2. Compute $a_{2^0} + a_{2^1} + \cdots + a_{2^{20}}$.

Answer: 21

Solution: We prove that $a_n = 1$ for all n by strong induction. Clearly $a_1 = 1$. Suppose that $a_n = 1$ for all values less than n. Let $n = 2^m + r$ for $0 \leq r < 2^m$. Consider $S = \{2^m, 2^m + 1, \dots, 2^m + r\}$. For each s in S, $\pi(s) + s$ has to be 2^{m+1}, since every other power of 2 is either too high or too low. Therefore, there is a unique spot determined for everything, by employing the inductive hypothesis. Finally, we note that $1048576 = 1024^2 = 2^{20}$, so we have exponents from 0 to 20, or 21 numbers.

4. Three identical balls are painted white and black, so that half of each sphere is a white hemisphere, and the other half is a black one. The three balls are placed on a plane surface, each with a random orientation, so that each ball has a point of contact with the other two. What is the probability that at at least one point of contact between two of the balls, both balls are the same color?

Answer: $\frac{95}{108}$

Solution: Note that the centers of the balls are the vertices of an equilateral triangle. Look at the cross section of a ball that lies in the plane of this triangle. The angle between the points of contact, as viewed from the center of the great circle of the ball in this plane, is $\frac{\pi}{3}$. The plane that divides the ball into the two hemispheres must intersect the cross section along a diameter. This line is determined by one point of intersection with the circle. There is a $\frac{2}{3}$ chance that the point is at least $\frac{\pi}{3}$ from one of the vertices, in the appropriate direction. We know calculate the probabiblity that each of the points of contact are different colors, or unmatched. For each ball, there are two states: the two points of contact from the ball are the same color, or the points of contact are different colors. Let S be the first case, and D be the second. All contact points are unmatched only when the configuration is DDD, SSD, or some permutation of those. In particular, all contact points are never unmatched when the configuration is SDD or some permutation. The probability of DDD is $\left(\frac{1}{3}\right)^3$ and SSD is $(\frac{2}{3}^2 \cdot \frac{1}{3}$, times 3 $= \frac{4}{9}$ to account for the permutation of which ball is in the D state. For the DDD configuration, after the first ball's orientation is fixed, there is a $\left(\frac{1}{2}\right)^2$ chance that the other two are oriented correctly. In the SSD case, fix one of the S balls. Then there is a $\frac{1}{2}$ chance the other S ball is oriented properly (so that the point of contact between the two S balls is unmatched). The D ball also has a $\frac{1}{2}$ chance of being in the proper orientation. Overall the probability that all contact points are unmatched is $\frac{1}{27} \cdot \frac{1}{4} + \frac{4}{9} \cdot \frac{1}{4} = \frac{1}{108} + \frac{12}{108} = \frac{13}{108}$. We are looking for the complement, $1 - \frac{13}{108} = \frac{95}{108}$

5. Compute the greatest positive integer n such that there exists an odd integer a, for which $\dfrac{a^{2^n} - 1}{4^{4^4}}$ is not an integer.

Answer: 509

Solution: Note that $4^{4^4} = 2^{512}$. We prove that for any $n \geq 3$, $a^{2^{n-2}} \equiv 1 \pmod{2^n}$. The base case $n = 3$ is trivial. Suppose the statement is true for any $n = k$. Then $a^{2^{k-2}} \equiv 1 \pmod{2^k}$, so there exists an integer b such that $a^{2^{k-2}} - 1 = 2^k b$. Rearranging and then squaring gives $a^{2^{k-2}} = 1 + 2^k b \Rightarrow a^{2^{k-1}} = 1 + 2^{k+1}(b + 2^{k-1}b^2) \equiv 1 \pmod{2^{k+1}}$, which completes the induction. Therefore, in the problem, $n = 512 - 2 = 510$ will make the expression be an integer. In adddition, $a^{2^k} \equiv 1 \pmod{2^512} 2^{512} | (a^{2^k} - 1) \Rightarrow 2^{512} | (a^{2^k} - 1) \cdot (a^{2^k} + 1) = a^{2^{k+1}} - 1 \Rightarrow a^{2^{k+1}} \equiv 1 \pmod{2^512}$, so this will hold for all higher $n \geq 510$.
All that is left to show is that for at least one a, $a^{2^{509}} - 1$ is not divisible by 2^{512}. We prove this in the case $a = 3$. Define $V_2(k)$ to be the highest power of 2 that divides k. We claim that $V_2(3^{2^k} - 1) = k + 2$ for all $k \geq 1$. When $k = 1$, $V_2(3^2 - 1) = v_2(2^3) = 3 = 1 + 2$. Suppose $V_2(3^{2^{k-1}} - 1) = k + 1$ for some $k \geq 2$. By the difference of squares factorization, $3^{2^k} - 1 = (3^{2^{k-1}} - 1)(3^{2^{k-1}} + 1)$, so $V_2(3^{2^k} - 1) = V_2(3^{2^{k-1}} - 1) + V_2(3^{2^{k-1}} + 1) = k + 1 + V_2(3^{2^{k-1}} + 1)$ by the inductive hypothesis. Because $k \geq 2$, $3^{2^{k-1}} + 1 \equiv 2 \pmod{8}$, so $V_2(3^{2^{k-1}} + 1) = 1$. So we have $V_2(3^{2^k} - 1) = k + 2$, completing the induction. Therefore if $n = 509$, the highest power of 2 dividing $3^{2^{509}}$ is 511. Hence the expression is not an integer mod 512.

6. You are blind and cannot feel the difference between a coin that is heads up or tails up. There are 100 coins in front of you and are told that exactly 10 of them are heads up. On the back of this paper, explain how you can split the otherwise indistinguishable coins into two groups so that both groups have the same number of heads.

Answer: N/A

Solution: Separate into group of 10 and group of 90. Flip over all coins in group of 10.

7. On the back of this page, write the best math pun you can think of. You'll get a point if we chuckle.

Answer: N/A

Solution: To be decided by scoring room.

8. Pick an integer between 1 and 10. If you pick k, and n teams total pick k as well, then you'll receive $\frac{k}{10n}$ points.

Answer: N/A

Solution: To be calculated by scoring room.

9. There are four prisoners in a dungeon. Tomorrow, they will be separated into a group of three in one room, and the other in a room by himself. Each will be given a hat to wear that is either black or white – two will be given white and two black. None of them will be able to communicate with each other and none will see his or her own hat color. The group of three is lined up, so that the one in the back can see the other two, the second can see the first, but the first cannot see the others. If anyone is certain of their hat color, then they immediately shout that they know it to the rest of the group. If they can secretly prove it to the guard, they are saved. They only say something if they're sure. Which person is sure to survive?

Answer: second in line

Solution: If the third person in line sees two white, then he knows he is wearing black and will call it out. If he doesn't say something, the second person knows he is different from the one in front of him.
10. Down the road, there are 10 prisoners in a dungeon. Tomorrow they will be lined up in a single room and each given a black or white hat – this time they don't know how many of each. The person in the back can see everyone's hat besides his own, and similarly everyone else can only see the hats of the people in front of them. The person in the back will shout out a guess for his hat color and will be saved if and only if he is right. Then the person in front of him will have to guess, and this will continue until everyone has the opportunity to be saved. Each person can only say his or her guess of "white" or "black" when their turn comes, and no other signals may be made. If they have the night before receiving the hats to try to devise some sort of code, how many people at a minimum can be saved with the most optimal code? Describe the code on the back of this paper for full points.

Answer: 9

Solution: If the person at the back says black first, it means there are an odd number of white hats. If he says white, he sees an even number of white hats.

11. A few of the problems on this mixer contest were taken from last year's event. One of them had fewer than 5 correct answers, and most of the answers given were the same incorrect answer. Half a point will be given if you can guess the number of the problem on this test that corresponds to last year's question, and another $.5$ points will be given if you can guess the very common incorrect answer.

Answer: 4, $\frac{7}{8}$

Solution: There were about 2 correct answers. Almost everyone put $\frac{7}{8}$.