Math Majors of America Tournament for High Schools

2015 Individual Test Solutions

1. The sum of two numbers, $a+b$, is 4 more than their difference, $a-b$. What is b?

Answer: 2

Solution: If a and b are the numbers, then $(a+b) = 4 + (a-b)$, so $b = 2$.

2. Rectangle $ABCD$ has $AB = CD = \sqrt{3}$ and $AD = BC = 1$. Line ℓ is the perpendicular bisector of $\overline{AC}$ and intersects $\overline{AC}$ and $\overline{AB}$ at P and Q, respectively. What is the area of quadrilateral $PQBC$?

Answer: $\frac{\sqrt{3}}{3}$ **OR** $\frac{1}{\sqrt{3}}$

Solution: Note that the diagonal creates a 30-60-90 triangle. By the Pythagorean theorem, $AC = 2$, so $PC = 1$. Since APQ is 30-60-90, we know that $PQ = \frac{\sqrt{3}}{3}$. Hence $QB = \frac{\sqrt{3}}{3}$. We calculate the area of CPQ and CBQ separately, and then add the two to get $\frac{\sqrt{3}}{3}$

3. The polynomial $p(x) = x^3 + cx - 2$ has roots that are all integers. What is c?

Answer: -3

Solution: Since there is no x^2 term, the sum of the roots is 0. Since 2 is prime, two of the roots are 1 and the other is -2.

4. There are 10 balls in a bucket, and there are 5 colors. Each color has exactly 2 balls of that color. Every time a ball is selected uniformly, randomly, and independently from the bucket, its color is noted and the ball is replaced. What is the expected number of selections from the bucket until one ball of every color has been seen?

Answer: $\frac{137}{12}$

Solution: Recall that in any geometric distribution, if the probability of the even is p, then the expected number of trials until that event occurs is $\frac{1}{p}$. Let $E(k)$ be the number of expected draws to choose k different colors. Clearly $E(1) = 1$. Then the probability of choosing a ball of a different color is $\frac{8}{10}$. So $E(2) = E(1) + \frac{10}{8}$. Now the probability of a new color is $\frac{6}{10}$, so $E(3) = E(2) + \frac{10}{6}$. Continuing recursively, it can be seen that $E(5) = E(1) + \frac{10}{8} + \frac{10}{6} + \frac{10}{4} + \frac{10}{2} = \frac{137}{12}$.

5. Consider a solid rectangular prism with length 10, width 8, and height 6. Find the volume of the set of points that are both a distance of at most 3 from the prism and a distance of at least 1 from the prism.

Answer: $752 + (680/3)\pi$

Solution: The volume can be found by subtracting the volume of the region of points at most a distance of 1 from the prism from the volume of the region of points at most a distance of 3 from the prism. The volume of the region of points at most a distance of 3 from the prism $= 10*8*6 + 2*3*(80+48+60) + 4*\frac{1}{4}*9\pi*(10+8+6) + (4/3)*27\pi = 1608 + 252\pi$. The volume of the region of points at most a distance of 1 from the prism $= 10*8*6 + 2*1*(80+48+60) + 4*\frac{1}{4}*\pi*(10+8+6) + (4/3)*\pi = 856 + (76/3)\pi$. The difference between the two is $752 + (680/3)\pi$.

6. Two positive integers, a and b are chosen randomly, uniformly, and independently from the set of positive integers less than 1000, $\{1, 2, \ldots, 1000\}$. What is the expected value of the number of quadrants through which the graph $x^a + y^b = 1$ will pass?

Answer: 3.75 **or** $\frac{15}{4}$

Solution: Without loss of generality suppose that a (or b, respectively) is even. Then (x, y) is a solution if and only if $(-x, y)$ is a solution. Now we show that y can be positive or negative. Note that

$$y = \sqrt[b]{1 - x^a}.$$

If b is odd, then as long as x is large enough, the expression will be negative. If b is even, then symmetry means that $-\sqrt[b]{1-x^a}$ is a solution whenever $\sqrt[b]{1-x^a}$ is a solution.

7. John and Mitchell are playing a game to see who gets the last of their candy. John rolls three unbiased 7-sided dice with sides labeled 1 through 7 and records the maximum roll. Mitchell flips a biased coin (with probability of heads p) 10 times and records the number of heads. What is the smallest p such that Mitchell has 50% or greater chance of winning the candy by recording a larger number?

Answer: 4/7

Solution: Compute expected value of the maximum of these rolls. This is 40/7. One direct way to calculate this is via the sum $\sum_{k=1}^{7} k(\frac{k^3-(k-1)^3}{7^3}) = 40/7$. Since the expected value for Mitchell is np, he needs to get 40/7 expected heads, or $p = \frac{4}{7}$, in 10 coin tosses.

8. Compute the absolute minimum of the function

$$f(x) = \cos(2x) + 3\cos(x).$$

Answer: $-\frac{17}{8}$

Solution: Because $\cos(2x) = 2\cos^2(x) - 1$, we have

$$\begin{aligned}\cos(2x) + 3\cos(x) &= 2\cos^2(x) - 1 + 3\cos(x)\\ &= \frac{1}{8}\left(16\cos^2(x) - 8 + 24\cos(x)\right)\\ &= \frac{1}{8}\left(16\cos^2(x) + 24\cos(x) + 9\right) - \frac{17}{8}\\ &= \frac{1}{8}\left(4\cos(x) + 3\right)^2 - \frac{17}{8}\end{aligned}$$

Since the square is always nonnegative, it can be no less than $-\frac{17}{8}$. Since $|\frac{3}{4}| < 1$, there is a value of cosine for which the first term vanishes, and this is indeed the minimum.

9. Define the sum of a finite set of integers to be the sum of the elements of the set. Let D be the set of positive divisors of 700. How many nonempty subsets of D have an even sum? (Simplify as reasonably as possible)

Answer: $2^{17} - 1$

Solution: Writing $700 = 2^2 \cdot 5^2 \cdot 7^1$, we see that the number of positive divisors of 700 is $(2+1)(2+1)(1+1) = 18$. Because D has 18 elements, it has 2^{18} subsets. For any subset S of D, let

$$f(S) = \begin{cases} S \cup \{1\}, & \text{if } 1 \notin S;\\ S \backslash \{1\}, & \text{if } 1 \in S.\end{cases}$$

The function f is an involution that maps subsets with even sums to subsets with odd sums and vice versa. This shows that exactly half of the 2^{18} subsets of D have even sums. Hence, the answer is 2^{17}, but we subtract 1 since we need to get rid of the case of the empty set.

10. The largest prime factor of $520,302,325$ has 5 digits. What is this prime factor?

Answer: 10613

Solution: We quickly notice that 520302325 is divisible by 5, resulting in the quotient 104060465. Using the Sophie Germain Identity, we can rewrite this as $(100+1)^4 + 4 \cdot 2^4 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2) = 9805 \cdot 10613$. Thus, the 5-digit prime factor is $\boxed{10613}$.

11. We play the following game with an equilateral triangle of $\frac{n(n+1)}{2}$ coins (with n coins on each side). Initially, all of the coins are turned heads up. On each turn, we may turn over three coins that are mutually adjacent; the goal is to make all of the coins eventually turned tails up. What is the 7^{th} smallest positive n for which this can be achieved?

Answer: 11

Solution: This can be done only for all $n \equiv 0, 2 \pmod 3$. Below by a triangle, we will mean three coins which are mutually adjacent. For $n = 2$; clearly it can be done and for $n = 3$ turn each of the four triangles. For $n \equiv 0, 2 \pmod 3$ and $n > 3$ turn every triangle. Then the coins at the corners are flipped once. The coins on the sides (not corners) are flipped three times each. So all these coins will have tails up. The interior coins are flipped six times each and have heads up. Since the interior coins have side length $n \equiv 2, 3$, by the induction step, all of them can be flipped so to have tails up. Next suppose $n \equiv 1 \pmod 3$: Color the heads of each coin red, white and blue so that adjacent coins have different colors and any three coins in a row have different colors. Then the coins in the corner have the same color, say red. A simple count shows that there is one more red coin than white or blue coins. So the (odd or even) parities of the red and white coins are different in the beginning. As we flip the triangles, at each turn, either (a) both red and white coins increase by 1 or (b) both decrease by 1 or (c) one increases by 1 and the other decreases by 1. So the parities of the red and white coins stay different. In the case all coins are tails up, the number of red and white coins would be zero and the parities would be the same. So this cannot happen.

12. Let $S = \{a_1, a_2, a_3, \ldots, a_{2015}\}$ be the first 2015 positive integers that a can be so that $2 + 2\sqrt{28a^2 + 1}$ is an integer. Compute the number of elements in S that a can be so that $2 + 2\sqrt{28a^2 + 1}$ is *not* a perfect square.

Answer: 0

Solution: We claim that this is always a square whenever it's an integer. If it's an integer, then it has to be even; so set it as $2k$; then simplifying, squaring and simplifying again eventually gives $k(k-2) = 28a^2$. Clearly k is even, so let $k = 2b$, which gives us $b(b-1) = 7a^2$.
Then since $\gcd(b, b-1) = 1$, we have two cases: either $b = 7x^2$, $b - 1 = y^2$, or $b = x^2, b - 1 = 7y^2$ (for integers x, y).
In the second case, we have that the whole thing is $2k = 4b = 4x^2 = (2x)^2$, so we're done.
In the first case, we have that $1 = b - (b-1) = 7x^2 - y^2$, but this is impossible mod 7, as -1 is not a quadratic residue (as $7 \equiv 3 \pmod 4$). Hence this case is impossible, and the whole thing must be a square.