MMATHS 2024 Tiebreaker Round Solutions

Yale Math Competitions

October 2024

1. Let f be a function over the domain of all positive real numbers such that

$$f(x) = \frac{1-\sqrt{x}}{1+\sqrt{x}}$$

Now, let g(x) be the function given by

$$g(x) = f(x)^{\frac{2f(\frac{1}{x})}{f(x)}}$$

g(100) can be expressed as a fraction

$$\frac{a}{b}$$

where a and b are relatively prime integers. What is the sum of a and b?

Proposed by: Noah Beckish

Answer: $\boxed{202}$

Solution: $f(1/x) = -f(x)$, so $g(x)$ can be simplified to $f(x)^{-2}$. $g(100) = f(100)^{-2} = (-9/11)^{-2} = 121/81$, so $a = 121$, and $b = 81$.

2. Define the *factorial function* of n, denoted $\partial(n)$, as the sum of the factorials of the digits of n. For example, $\partial(2024) = 2! + 0! + 2! + 4! = 29$. There are four positive integers such that $\partial(\partial(n)) = n$ and $\partial(n) \neq n$. Given that $n = 871$ is one of them, compute the sum of the other three.

Proposed by: Owen Zhang

Answer: $\boxed{91595}$

Solution: Check that $n = 872$ also works ($\partial(872) = \partial(871) + 1 = 45362$). Also, we must have that $\partial(871)$ and $\partial(872)$ also work, since $\partial(\partial(871)) = 871$ and $\partial(\partial(872)) = 872$. Then we have:

$$872 + \partial(871) + \partial(872) = 91595$$