MMATHS 2024 Team Round Solutions

Yale Math Competitions

October 2024

1. On a planet, far far away, the Yaliens have defined: x ""equals"" y if and only if $|x - y| \leq 3$. Let S be a set of positive integers. What is the smallest possible number of elements in S such that, for any positive integer r, where $1 \leq r \leq 2024$, r ""equals"" some element in S?

   Proposed by: Noah Ripke

   Answer: $\boxed{290}$

   Solution: Take for example $x = 2021$, then the set of all y that "equal" x is $y \in [2018, 2024]$. The minimum possible number of elements in S will occur when r usually only "equals" one element in S. To do this, choose $x_1 = 2021$, $x_2 = 2014$, $x_3 = 2007$, ... $x_{289} = 5$ to be in S. This finally leaves $r = 1$ which doesn't equal any element in S so put 1 in S so S has 290 elements.

2. Consider the recursive sequence defined by $a_{n+1} = a_n^n + 1$, with $a_1 = 0$. What is the last digit of a_{2024}?

   Proposed by: Howard Dai

   Answer: $\boxed{1}$

   Solution: The first few numbers are as such:

   $$0, 1, 2, 9, 2, 3, 0, 1, 2, 3, 0, 1, \ldots$$

   Note that once we have a cycle of four: 2,3,0,1. Because the last digit of any 2^n is equivalent to the last digit of 2^{n+4} (same for 3,0,1), we see that this cycle will infinitely repeat. Then because 2024 is divisible by 4, the last digit of a_{2024} will be 1.

3. Alice picks a random three digit number, from 100 to 999, inclusive. The probability that her first digit is larger than the sum of her other two digits can be expressed as a common fraction $\frac{a}{b}$. Find $a + b$.

   Proposed by: Zubin Mukerjee

   Answer: $\boxed{71}$

   Solution: There are 900 possible choices, of which $1 + 3 + 6 + 10 + \cdots + 45 = 165$ satisfy the condition. This result can be derived as the ninth tetrahedral number, as the sum of the first nine triangular numbers, or simply by casework. The final answer will be 165/900, which reduces to 11/60.

4. Consider a pattern of squares and triangles. The first move of the pattern is to place an isosceles right triangle with side lengths $1, 1, \sqrt{2}$. For every subsequent move, you need to attach a square to every non-hypoteneuse side of a triangle and attach the same isosceles right triangle to every side of a square. After 2024 moves, what is the smallest possible area of the resulting shape?

   Proposed by: Noah Ripke

   Answer: $\boxed{4047}$

   Solution: The pattern which will minimize area, will minimize the amount of triangles and squares you add each move. This means that whenever you can, orient the triangle placement so it covers two open sides of a square. Then place triangles on every open square face, so that it only takes 1 square to cover both of the legs. Just this rule will force the pattern into a repeating sequence forming a diagonal rectangle. In general, we will find that for area after move n is $A = 2n - 1$. So for move 2024, we get the total area to be 4047.

5. Amir and Bella play a game on a gameboard with 6 spaces, labeled 0, 1, 2, 3, 4, and 5. Each turn, each player flips a fair coin. If it is heads, their character moves forward one space, and if it is tails, their character moves back one space, unless it was already at space 0, in which case it moves forward one space instead. If Amir and Bella each have a character that starts at space 0, what is the probability that they end turn 5 on the same space?

   Proposed by: Henry Burton

   Answer: $\boxed{191}$

   Solution: The key is to see the bijection to a game where squares are numbered from -5 to 5, and at every square, the character has an equal chance of moving forward or backwards. At the end, you only consider the absolute value of each character's square.

   The number of cases where both characters end on square $\pm 5 = 4 \cdot ((5C5)^2)$. For ± 3, you get $4 \cdot ((5C4)^2)$ cases, and for ± 1, you get $4 \cdot ((5C3)^2)$. These add up to $4 \cdot 126$ cases, out of $2^1$0total, which simplifies to $\frac{63}{128}$.

6. Cat and Claire are having a discussion about their favorite positive two-digit numbers.

   Cat: My number has a 1 in its tens digit. Is it possible that your number is a multiple of my number?"

   Claire: No, however, my number is not prime. Additionally, if I told you the two digits of my number, you still wouldn't know my number.

   Cat: Aha, my number and your number aren't relatively prime!

   Claire: Then our numbers must share the same ones digit!

   What is the product of Cat and Claire's numbers?

   Proposed by: Benjamin Wu

   Answer: $\boxed{1316}$

   Solution: Based on Cat's first clue, and Claire's information on her number not being prime, Claire's number must be a number that is some number less than 10 times a prime number greater than 20, or products of 3s, 7s, and other values. There are about 20 numbers that aren't prime and aren't a multiple of any number from 10-19, but the only pair that shares the same digits is 49 and 94 (this is the only way Cat wouldn't know Claire's number if she told her the digits). The only number that Cat could have that isn't relatively prime to either number is 14, and therefore Claire's number is 94 because it has the same ones digit.

7. Bill has the expression $1 + 2 + 3 + \cdots + 8$. He replaces two different addition symbols with multiplication symbols uniformly at random. What is the average value that he obtains?

   Proposed by: Stephen Xia

   Answer: $\boxed{1912}$

   Solution: Note that there are $\binom{7}{2} = 21$ different ways to insert the two multiplication symbols. Now, we can find the sum of all 21 different possible expressions by separately counting the number of times each number is added individually and the number of times each possible product is added. We first focus on multiplication and analyzing the number of times possible products are added.

   Case 1: Consecutive Multiplication Signs
   We note that the products are $(n)(n+1)(n+2)$ for $n = 1$ to $n = 6$, and doing the quick arithmetic gives a sum of $6 + 24 + 60 + 120 + 210 + 336 = 756$.

   Case 2: Non-Consecutive Multiplication Signs
   In this case, we can count the number of times each product $(n)(n+1)$ appears when the multiplication signs are not in adjacent spots. Note that 1×2 and 7×8 each appear 5 times and the others all appear 4 times, noting that the products on the ends appear an extra time because 1 and 8 are only adjacent to one sign. Therefore, we have the sum $2*5 + 6*4 + 12*4 + 20*4 + 30*4 + 42*4 + 56*5 = 730$.

   Now, we need to figure out how many times each number is "added" via addition rather than included in a product across all cases, and we will have our final sum. Note that for 1 and 8 to not be in multiplication, the multiplication signs can be placed across the other six spots since 1 and 8 are only adjacent to one sign, which means they are added $\binom{6}{2} = 15$ times. For the other numbers, they are adjacent to two signs, so the multiplication signs can be placed across the other five spots so they are added $\binom{5}{2} = 10$ different times. This means that the "addition parts" of these sums sum to $1*15+2*10+3*10+4*10+5*10+6*10+7*10+8*15 = 405$.

   This means that the total sum of all 21 expressions is $756 + 730 + 405 = 1891$, so the average value is $\frac{1891}{21}$, and hence our answer is $1891 + 21 = \mathbf{1912}$.

8. Triangle ABC is an acute triangle with $BC = 6$ and $AC = 7$. Let D, E, and F be the feet of the altitudes from A, B, and C respectively. $\overline{AD}$ bisects angle FDE. Let m be the maximum possible value of $FD + ED$. Find m^2.

   Proposed by: Brady Exoo

   Answer: $\boxed{36}$

   Solution: Reflect A, E, F across BC to A', E', F', producing a cyclic hexagon $ECE'F'BF$. Call the circle this hexagon is inscribed in O. Note that BC is the diameter O, so no chord can have length greater than BC. Also note that F, D, E' are collinear, as we must have $\angle A'DE' \cong \angle ADE \cong \angle ADF$, making $\angle ADF, \angle A'DE'$ vertical angles over line AA' (A, D, A' are collinear because $\overline{AD} \perp \overline{BC}$). Then $FD + DE = FD + DE' = FE'$, which is just the length of a chord to O. Thus, $FE' \leq BC = 6$. We can achieve $FE' = 6$ if D is the center of O, which occurs when $BD = DC$. Then setting $AB = AC = 7$ produces an isosceles triangle, where AD bisects BC as desired.

9. Grant and Stephen are playing Square-Tac-Toe. In this game, players alternate placing X's and O's on a 3x3 board, and the first person to complete a 2x2 square with their respective symbol wins the game. If all tiles are filled and no such square exists, the game is a tie. Grant moves first. Given that Stephen plays randomly and Grant plays optimally (knowing that Stephen is playing randomly), the probability that Grant wins is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.
(Note: Grant playing "optimally" means he is maximizing his win probability)

Proposed by: Howard Dai

Answer: $\boxed{29}$

Solution: There are four possible 2x2 squares which can be completed. We can call a 2x2 "blocked" if at least one of its tiles has been taken by Stephen, and Grant "contributes" to a 2x2 by taking a tile in that 2x2. In general, Grant's strategy can be captured by trying to maximize the number of unblocked 2x2 regions he contributes to on every move. We argue that all game sequences will end up in one of the following two scenarios before Stephen's third move.

SCENARIO 1: Grant has three X's in one unblocked 2x2, with all other 2x2 regions blocked. For example, here is a possible arrangement which falls into the category of scenario 1:

	O	
O	X	X
	X	

In this situation, Grant wins if and only if Stephen does not randomly block Grant's final X, which happens $\frac{3}{4}$ of the time.

SCENARIO 2: Grant has three X's in one unblocked 2x2, but there is another unblocked 2x2 region. For example, here is a possible arrangement:

O		
O	X	X
	X	

Here, if Stephen does not block Grant's last square, Grant wins immediately. Otherwise, if Stephen blocks, Grant can still place an X in the other unblocked 2x2, and Stephen will block the final square with probability $\frac{1}{2}$. So, in this situation, Grant wins with probability $\frac{3}{4} + \frac{1}{4} \cdot \frac{1}{2} = \frac{7}{8}$.

For the first move, it is clear the Grant should play in the center. This is because the center contributes to all four of the 2x2 squares, and any other play contributes strict subset of this.

If Stephen plays in any edge on the second move, Grant plays in the opposite edge:

	O	
	X	
	X	

In this specific example, if Stephen plays in the top row, Grant can place an X in one of the other two slots to create SCENARIO 2, winning with probability $\frac{7}{8}$. Otherwise, Stephen will block another 2x2, and Grant goes for the last 2x2, creating SCENARIO 1.

S2	O	S2
S1	X	S1
S1	X	S1

So, the probability that Grant wins given that Stephen plays in an edge is $\frac{1}{3} \cdot \frac{7}{8} + \frac{2}{3} \cdot \frac{3}{4} = \frac{19}{24}$.

If Stephen plays in any corner on the second move, Grant plays in one of the opposite edges:

O		
	X	
	X	

In this situation, if Stephen plays in any of the spots labelled S2, he will leave two adjacent unblocked 2x2 slots, which allows Grant to enter SCENARIO 2:

O	S2	S2
S2	X	S1
S2	X	S1

Thus, the probability that Grant wins given that Stephen plays in a corner is $\frac{2}{3} \cdot \frac{7}{8} + \frac{1}{3} \cdot \frac{3}{4} = \frac{20}{24}$.

Because Stephen plays an edge or corner on his second move with equal probability, the overall probability that Grant wins is $\frac{1}{2} \cdot \frac{19}{24} + \frac{1}{2} \cdot \frac{20}{24} = \frac{13}{16}$.

10. In acute $\triangle ABC, AB = 11$ and $CB = 10$. Points E and D are constructed such that $\angle CBE$ and $\angle ABD$ are right, and $ACEBD$ is a non-degenerate pentagon. Additionally, $\angle AEB \cong \angle DCB$, $AE = CD$ and $ED = 20$. Given that EA and CD intersect at P and $AP = 4$, find CP^2

Proposed by: Henry Burton

Answer: $\boxed{26}$

Solution: First note that because $\angle AEB \cong \angle DCB, AE = DC$, and $m\angle CBD = 90° + m\angle ABC = m\angle EBA$, we have $\triangle AEB \cong \triangle DCB$. Then we have $DB = AB = 11$ and $EB = CB = 10$. Now, notice that $\angle EBA$ and $\angle CBA$ are supplementary. Performing law of cosines on $\triangle EBD$:

$$20^2 = 10^2 + 11^2 - 2 \cdot 11 \cdot 10 \cdot \cos(\angle EBD)$$

Performing law of cosines on $\triangle ABC$:

$$AC^2 = 10^2 + 11^2 - 2 \cdot 11 \cdot 10 \cdot \cos(180 - \angle EBD)$$

$$AC^2 = 10^2 + 11^2 + 2 \cdot 11 \cdot 10 \cdot \cos(\angle EBD)$$

Then we have:

$$AC^2 + 20^2 = 2(10^2 + 11^2)$$
$$AC^2 = 42$$

Also, we can notice that $\triangle AEB$ is a 90 degree rotation of $\triangle DCB$, so $\overline{DC}$ and $\overline{AE}$ must be perpendicular. Thus, $\triangle APC$ is a right triangle, so we have:

$$42 = 4^2 + CP^2$$
$$CP^2 = 26$$

11. Define a sequence $a_{m,n}$ where $a_{m,0} = 1$, and for all other m, n (assuming $m \geq 1$):

$$a_{m,n} = \begin{cases} 0 & n < 0 \\ 1 & n \equiv 0 \mod m \\ a_{m,n-1} + a_{m,n-m} & else \end{cases}$$

If $\frac{a_{2025,2025^2-1}}{a_{2024,2024^2-1}} = \frac{a}{b}$, then what is $a + b$?

Proposed by: Noah Ripke

Answer: 5059

Solution: This problem seems intimidating, so we try to break it down into smaller parts. We can notice that if you set m to be constant, then this uniquely determines a sequence over n. The notice that the sequence seems to "restart" in a way, every time $n \equiv 0 \mod m$. This, and seeing that there is a square number in the desired answer, we can consider organizing the sequence as a square. Take $m = 4$ for example:

$$a_{4,0}, a_{4,1}, a_{4,2}, a_{4,3}$$
$$a_{4,4}, a_{4,5}, a_{4,6}, a_{4,7}$$
$$a_{4,8}, a_{4,9}, a_{4,10}, a_{4,11}$$
$$a_{4,12}, a_{4,13}, a_{4,14}, a_{4,15}$$

We see that the number in the sequence we want is the bottom right square. We also notice that everything in the top row is 1, and everything on the leftmost column is 1. Every other element is the sum of the element above it, and the element to the left of it. What this creates is pascal's triangle in a square form! Indeed, the element in the bottom right corner is in the $(2m-2)$th row of pascal's triangle, and is in the exact middle. Thus we find that $a_{m,m^2-1} = \binom{2m-2}{m-1}$. Thus we want: $\frac{\binom{4048}{2024}}{\binom{4046}{2023}} = \frac{4047 \cdot 4048}{2024 \cdot 2024} = \frac{4047}{1012}$ thus $a + b = 4047 + 1012 = 5059$.

12. $S_1, S_2, \ldots, S_n$ are subsets of $\{1, 2, \ldots, 10000\}$ which satisfy that, whenever $|S_i| > |S_j|$, the sum of all elements in S_i is less than than the sum of all elements in S_j. Let m be the the maximum number of distinct values among $|S_1|, \ldots, |S_n|$. Find $\lfloor \frac{m}{100} \rfloor$.

Proposed by: Kelin Zhu (UMD)

Answer: 41

Solution: Let $f(S)$ denote the sum of elements of the set S. Suppose there is only one subset of each size. Also, suppose $|S_1| < \ldots < |S_n|$. We can change S_n to the form $\{1, 2, \ldots, b\}$ and S_1 to the form $\{a, \ldots, 2024, 2025\}$, which would increase $f(S_1)$ and decrease $f(S_n)$, preserving the inequalities $f(S_1) > f(S_2) > \ldots > f(S_n)$.

Now, note that if we fix $|S_i|$, $f(S_i)$ can take any integer value between $1 + 2 + \ldots + |S_i|$ and $n+(n-1)+\ldots+(n-|S_i|+1)$. This means that we only need to ensure that $|S_n|-|S_1| \geq n-1$, and since $\min f(S_i) < f(S_n), \max f(S_i) > f(S_1)$, for all $2 \leq i \leq n-1$, we can choose $S_2, \ldots, S_{n-1}$ so that $f(S_1) > f(S_2) > \ldots > f(S_n)$ as long as $f(S_1) - f(S_n) \geq n-1$. Also, this means that we can assume $|S_i| + 1 = |S_{i+1}|$, so that $n = |S_n| - |S_1| + 1$.

Now, for each value of $|S_1|$, consider the maximum possible value of $|S_n|$. When $|S_1|$ changes from a to $a+1$, $|S_n|$ changes from b to $b+1$, if $a > b$, the value $f(S_1) - f(S_n)$ is increased and we might be able to increase $|S_n|$ further. If $a < b$, the value of $f(S_1) - f(S_n)$ is decreased and we won't get a be able to increase $|S_n|$ further. Hence, the maximum value of n is obtained at the largest value of $|S_1|$ where $a \geq b$.

At these values of a and b, we have $1 + 2 + \ldots + b \approx \frac{1+2+\ldots+2025+a-b}{2}$. Since $a - b$ is insignificant compared to $1 + 2 + \ldots + 2025$, $1 + 2 + \ldots + b \approx \frac{1+2+\ldots+2025}{2}$, so $b \approx \frac{2025}{\sqrt{2}}$. Then, $a - b + 1 \approx (\sqrt{2} - 1)2025 + 1$, and using any reasonable approximation of $\sqrt{2}$, we obtain the answer $\boxed{8}$.