MMATHS

Team: ______________________ **Team ID:** ___________

2024 Team Round

1. ________
2. ________
3. ________
4. ________
5. ________
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8. ________
9. ________
10. ________
11. ________
12. ________

– 45 minutes
– no calculators
– positive integer answers

1. On a planet, far far away, the Yaliens have defined: x "equals" y if and only if $|x-y| \leq 3$. Let S be a set of positive integers. What is the smallest possible number of elements in S such that, for any positive integer r, where $1 \leq r \leq 2024$, r "equals" some element in S?

2. Consider the recursive sequence defined by $a_{n+1} = a_n^n + 1$, with $a_1 = 0$. What is the last digit of a_{2024}?

3. Alice picks a random three digit number, from 100 to 999, inclusive. The probability that her first digit is larger than the sum of her other two digits can be expressed as a common fraction $\frac{a}{b}$. Find $a+b$.

4. Consider a pattern of squares and triangles. The first move of the pattern is to place an isosceles right triangle with side lengths $1, 1, \sqrt{2}$. For every subsequent move, you need to attach a square to every non-hypoteneuse side of a triangle and attach the same isosceles right triangle to every side of a square. After 2024 moves, what is the smallest possible area of the resulting shape?

5. Amir and Bella play a game on a gameboard with 6 spaces, labeled 0, 1, 2, 3, 4, and 5. Each turn, each player flips a fair coin. If it is heads, their character moves forward one space, and if it is tails, their character moves back one space, unless it was already at space 0, in which case it moves forward one space instead. If Amir and Bella each have a character that starts at space 0, the probability that they end turn 5 on the same space can be expressed as a common fraction $\frac{a}{b}$. Find $a+b$.

6. Cat and Claire are having a discussion about their favorite positive two-digit numbers.
Cat: My number has a 1 in its tens digit. Is it possible that your number is a multiple of my number?"
Claire: No, however, my number is not prime. Additionally, if I told you the two digits of my number, you still wouldn't know my number.
Cat: Aha, my number and your number aren't relatively prime!
Claire: Then our numbers must share the same ones digit!
What is the product of Cat and Claire's numbers?

7. Bill has the expression $1+2+3+\cdots+8$. He replaces two different addition symbols with multiplication symbols uniformly at random. The value that he obtains on average can be expressed as a common fraction $\frac{m}{n}$. Find $m+n$.

8. Triangle ABC is an acute triangle with $BC = 6$ and $AC = 7$. Let D, E, and F be the feet of the altitudes from A, B, and C respectively. $\overline{AD}$ bisects angle FDE. Let m be the maximum possible value of $FD + ED$. Find m^2.

9. Grant and Stephen are playing Square-Tac-Toe. In this game, players alternate placing X's and O's on a 3x3 board, and the first person to complete a 2x2 square with their respective symbol wins the game. If all tiles are filled and no such square exists, the game is a tie. Grant moves first. Given that Stephen plays randomly and Grant plays optimally (knowing that Stephen is playing randomly), the probability that Grant wins is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.
(Note: Grant playing "optimally" means he is maximizing his win probability)

10. In acute $\triangle ABC$, $AB = 11$ and $CB = 10$. Points E and D are constructed such that $\angle CBE$ and $\angle ABD$ are right, and $ACEBD$ is a non-degenerate pentagon. Additionally, $\angle AEB \cong \angle DCB$, $AE = CD$ and $ED = 20$. Given that EA and CD intersect at P and $AP = 4$, find CP^2.

11. Define a sequence $a_{m,n}$ where $a_{m,0} = 1$, and for all other m, n (assuming $m \geq 1$):

$$a_{m,n} = \begin{cases} 0 & n < 0 \\ 1 & n \equiv 0 \mod m \\ a_{m,n-1} + a_{m,n-m} & \textit{else} \end{cases}$$

If $\frac{a_{2025,(2025^2-1)}}{a_{2024,(2024^2-1)}} = \frac{a}{b}$ where a and b are relatively prime positive integers, then what is $a+b$?

12. $S_1, S_2, \ldots, S_n$ are subsets of $\{1, 2, \ldots, 10000\}$ which satisfy that, whenever $|S_i| > |S_j|$, the sum of all elements in S_i is less than than the sum of all elements in S_j. Let m be the the maximum number of distinct values among $|S_1|, \ldots, |S_n|$. Find $\lfloor \frac{m}{100} \rfloor$.