MMATHS 2023 Team Round Solutions

Yale Math Competitions

October 2023

1. Lucy has 8 children, each of whom has a distinct favorite integer from 1 to 10, inclusive. The smallest number that is a perfect multiple of all of these favorite numbers is 1260, and the average of these favorite numbers is at most 5. Find the sum of the four largest numbers.

   Proposed by: Stephen Xia and Michael Gao

   Answer: $\boxed{28}$

   Solution: This question asks us to pick 8 distinct integers from 1 to 10 such that their least common multiple is $1260 = 2^2 \cdot 3 \cdot 5 \cdot 7$. Therefore, 8 cannot be in our list, since that would cause the LCM to have a 2^3 term. We also cannot have 9 in our list, since that would introduce a 3^2 term. Therefore, the eight numbers are $1, 2, 3, 4, 5, 6, 7, 10$. The average of these numbers is 4.75, which is indeed below 5. The sum of the largest four numbers is $5 + 6 + 7 + 10 = \boxed{28}$.

2. 20 players enter a chess tournament in which each player will play every other player exactly once. Some competitors are cheaters and will cheat in every game they play, but the rest of the competitors are not cheaters. A game is *cheating* if both players cheat, and a game is *half-cheating* if one player cheats and one player does not. If there were 68 more half-cheating games than cheating games, how many of the players are cheaters?

   Proposed by: Dixon Miller

   Answer: $\boxed{8}$

   Solution: If there are n cheaters, then there are $\binom{n}{2} = \frac{n(n-1)}{2}$ *cheating* games. On the other hand, there are $n(20-n) = -n^2 + 20n$ *half-cheating* games. We want to solve $-n^2 + 20n - 68 = \frac{n(n-1)}{2}$. Simplifying gives $3n^2 - 41n + 136 = 0$, which can be factored as $(3n-17)(n-8) = 0$. Since n is an integer, we must have $n = \boxed{8}$.

3. There are 360 permutations of the letters in $MMATHS$. When ordered alphabetically, starting from $AHMMST$, $MMATHS$ is the nth permutation. What is n?

   Proposed by: Benjamin Wu

   Answer: $\boxed{173}$

   Solution: There are 5! / 2 = 60 permutations that start with each of the letters A, H, T, and S, so the first permutation that starts with an M would be 60 * 2 + 1 = 121 (after all the As and Hs). Then with the 5 remaining letters AHMST, there are 4! = 24 permutations that start with each letter, so the first permutation that starts with MM would be 120 + 24 * 2 + 1 = 169. After that, A is already in the right place, and THS is the 5th permutation of the letter H, S, and T, so the answer is 120 + 24 * 2 + 5 = $\boxed{173}$.

4. How many distinct real numbers x satisfy the equation $4\cos^3(x) + \sqrt{x} = 3\sin(x) + \cos(3x)$?

   Proposed by: Howard Dai

   Answer: $\boxed{6}$

   Solution: We can begin by simplifying this expression (expand $\cos(3x)$):

$$4\cos^3(x) + \sqrt{x} = 3\sin(x) + \cos(3x)$$
$$4\cos^3(x) + \sqrt{x} = 3\sin(x) + 4\cos^3(x) - 3\cos(x)$$
$$\sqrt{x} = 3(\sin(x) - \cos(x))$$

   We can look at the intersection between the $\sqrt{x}$ graph and the $3(\sin(x) - \cos(x))$ graph. Because we are observing the difference between $\sin(x)$ and $\cos(x)$, we can look at the intersection between some line $y - x = b$ and the unit circle to find maximum and minimum points. b is maximized at $\frac{3\pi}{4}$ and minimized at $\frac{7\pi}{4}$, so the difference $\sin(x) - \cos(x)$ is maximized and minimized at these points, fluctuating with a period of 2π. At $\frac{3\pi}{4}, 3(\sin(x) - \cos(x)) = 3\sqrt{2}$, so our graph "peaks" at $3\sqrt{2}$. We need to find how many "peaks" our square root graph crosses through before it grows above $3\sqrt{2}$.

   At $x = \frac{3\pi}{4}, \sqrt{\frac{3\pi}{4}} \approx \frac{3}{2}$, which is much less than $3\sqrt{2}$, so it definitely passes through our first "peak". At $x = \frac{11\pi}{4}$ (the next "peak"), $\sqrt{\frac{11\pi}{4}} \approx 3$, which we can still confidently say is less than $3\sqrt{2}$. At $x = \frac{19\pi}{4}, \sqrt{\frac{19\pi}{4}} \approx \sqrt{15}$, which is less than (although close to) $3\sqrt{2} = \sqrt{18}$.

   At $x = \frac{27\pi}{4}, \sqrt{\frac{27\pi}{4}} > \sqrt{6\pi} > \sqrt{18} = 3\sqrt{2}$, so our graph is above the peak at this point. The $\sqrt{x}$ graph crossed through 3 peaks, which creates two intersections each time, giving us 6 total intersections.

5. $\omega_A, \omega_B, \omega_C$ are three concentric circles with radii 2, 3, and 7, respectively. We say that a point P in the plane is *nice* if there are points A, B, and C on ω_A, ω_B, and ω_C, respectively, such that P is the centroid of $\triangle ABC$. If the area of the smallest region of the plane containing all nice points can be expressed as $\frac{a\pi}{b}$, where a and b are relatively prime positive integers, what is $a + b$?

   Proposed by: Ruben Carpenter

   Answer: $\boxed{149}$

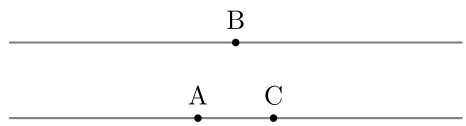
   Solution: By symmetry, we can see that each region will be circular in nature. Then it suffices to find the minimum and maximum radii for any nice point. Because a centroid is the average position of all three points, the farthest nice point (maximum radius) occurs when all three points are collinear and on the same "side". In this case, our point is $\frac{2+3+7}{3} = 4$ units away from the center. The closest nice point to the center (minimum radius) occurs when the points on circles with radii 2, 3 are opposite the point on the circle with radius 7 (to "pull" the point on the circle of radius 7 closer to the center). In this case, we have an average position of $\frac{7-2-3}{3} = \frac{2}{3}$ from the center, so the minimum radius is $\frac{2}{3}$. Then the area of this region is $\pi(4^2 - (\frac{2}{3})^2) = \frac{140}{9}$. Thus, we have $140 + 9 = \boxed{149}$.

6. 10 points are drawn on each of two parallel lines. What is the largest number of acute triangles of positive area that can be formed using three of these 20 points as vertices?

   Proposed by: Ruben Carpenter

   Answer: $\boxed{330}$

   Solution: First note that as long as the points are sufficiently "close" (or alternatively, the lines are sufficiently far), any configuration of some point B on one line located between the two points A, C on the other line will result in an acute triangle:



   Intuitively, alternating positions on each line will maximize these configurations. Then each point on one line can be bijectively mapped with an "adjacent" point on the other line, and counting the number of acute triangles can then be reduced to choosing points on one of the lines: for any choice of three points on one of the lines, replace the middle point with its corresponding point on the other line. Note that we also have cases where two of the points can be chosen as the same (because one of them is mapped to a corresponding point). To account for this, the problem is equivalent to choosing three unique points from $n+1$ points (and translating the rightmost point to the left by 1). We can perform this procedure for each side, giving us $\frac{2(n+1)(n)(n-1)}{6} = \frac{(n+1)(n)(n-1)}{3}$ acute triangles given n points on each side. In our specific problem, $n = 10$, so we have $\frac{(11)(10)(9)}{3} = \boxed{330}$ total acute triangles.

7. $ABCD$ is a regular tetrahedron of side length 4. Four congruent spheres are inside $ABCD$ such that each sphere is tangent to exactly three of the faces, the spheres have distinct centers, and the four spheres are concurrent at one point. Let v be the volume of one of the spheres. If v^2 can be written as $\frac{a}{b}\pi^2$, where a and b are relatively prime positive integers, find $a + b$.

   Proposed by: Jason Wang

   Answer: $\boxed{35}$

   Solution: By symmetry, the point of concurrency must be located at the centroid of the tetrahedron. Now it suffices to find the volume of a sphere which touches three sides and intersects the centroid. Without loss of generality, we can focus on the sphere which touches all sides containing point A. Let the centroid of the tetrahedron be O, and let O' be the centroid of face $\triangle ABC$. Note that by symmetry, the center of this sphere will lie on segment $\overline{AO}$. Let the center of the sphere be P, and let its perpendicular projection to face $\triangle ABC$ be P'. Then $\triangle AOO' \sim \triangle APP'$. Additionally, by the properties of a sphere, we must have $PP' = OP = r$. By triangle similarity, we have:

$$\frac{AO}{OO'} = \frac{AP}{PP'}$$

We can solve for OO' and AO directly; we know $OO' = \frac{1}{4}O'D$ (to show this, think about center of mass!). The height of $\triangle ABC$ is $2\sqrt{3}$, so $AO' = \frac{4\sqrt{3}}{3}$, and thus $O'D = \frac{4\sqrt{6}}{3}$. So, $OO' = \frac{\sqrt{6}}{3}$. We also have $AO = \frac{3}{4}O'D$, so $AO = \sqrt{6}$. Then we have:

$$\frac{\sqrt{6}}{\frac{\sqrt{6}}{3}} = \frac{AP}{PP'}$$
$$3 = \frac{AO - OP}{PP'}$$
$$3 = \frac{\sqrt{6} - OP}{PP'}$$
$$3 = \frac{\sqrt{6} - r}{r}$$
$$r = \frac{\sqrt{6}}{4}$$
$$v = \frac{4}{3}\pi r^3 = \frac{\sqrt{6}}{8}\pi$$
$$v^2 = \frac{3}{32}\pi^2$$

So, we have $a + b = 3 + 32 = \boxed{35}$

8. 30 people sit around a table, some of which are Yale students. Each person is asked if the person to their right is a Yale student. Yale students will always answer correctly, but non-Yale students will answer randomly. Find the smallest possible number of Yale students such that, after hearing everyone's answers and knowing the number of Yale students, it is possible to identify *for certain* at least one Yale student.

   Proposed by: Michael Gao

   Answer: $\boxed{22}$

   Solution: Note that we solve for the number of non-Yale students, Y.

   LOWER BOUND: First, call these 30 people $y_1, y_2, y_3, ..., y_30$. Further, define $F : 1, 2, ..., 30 \mapsto Y, N$ to be a function such that $F(i) = Y$ if y_i calls $y_(i+1)$ a Yale student, and N if vice versa. We consider the subsequences of consecutive Y's and N's, of lengths $Y_1, N_1, Y_2, N_2, ..., Y_m, N_m$ (notice that there must be an equal number of each type of sequence). We see the following:

   (a) If $F(a) = F(a+1) = F(a+2) = ... = F(a+b) = Y$, then we know either $y_(a+b+1)$ is a Yale student, or $y_a, y_(a+1), ..., y_(a+b+1)$ are all not Yale students.

   (b) If $F(a) = N$, then at least one of y_a and y_{a+1} are not Yale students. So, there must be at least: $(N_1/2) + (N_2/2) + ... + (N_m/2)$ non Yale students amongst those who said or were said to be non Yale students.

   Now, we are ready to begin the key observation. Inspect the maximum length string of Y's, let's arbitrarily assign as Y_1. We will show that the person at the end of this group is a Yale student.

We do so by contradiction. If the person at the end is not a Yale student, then by the first observation, all the people in this sequence are not Yale students. By the second observation, we can define $(N_1/2) + (N_2/2) + ... + (N_m/2) = k. Then, N_1 + N_2 + ... + N_m \leq 2k and m \leq k$. Thus, we can calculate the minimum length of Y_1, which is (30-2k)/k, simply since the maximum length cannot be less than the average length. Now, we express the number of non Yale students as at least $(30-2k)/k + k$, adding back at least k students given by the second observation. Realize $(30-2k)/k + k < 9$ is impossible as k is natural. Thus, the lower bound for $Y = 8$.

UPPER BOUND: consider the sequence YYYYNYYYYN...YYYYN. One possiblity is that everyone who is called Yale student is a Yale student, and everyone who is called not a Yale student is not a Yale student. However, for a block YYYYN, we can also switch it so the people who are called Yale students are actually not Yale students, and the person who is called not a Yale student is a Yale student. If so, realize that 5+4 = 9, meaning there are at most 9 non Yale students. In this configuration then, it is impossible to determine who is a Yale student. So, we must have $Y < 9$. Thus, we must have $Y = 8$. Then we have $30 - 8 = \boxed{22}$ Yale students.

9. Let $(x + x^{-1} + 1)^{40} = \sum_{i=-40}^{40} a_i x^i$. Find the remainder when $\sum_{p \text{ prime}} a_p$ is divided by 41.

Proposed by: Neil

Answer: $\boxed{4}$

Solution: Let p be a prime number. Let $(x+x^{-1}+1)^p$ have coefficients b_i for x^i. It's obvious to see that b_0, b_p, b_{-p} are all $1 \pmod p$. Also for any $k < p$, we have

$$b_k = \binom{p}{k} + \binom{p}{2}\binom{p-2}{k} + \binom{p}{4}\binom{p-4}{k} + ... + \binom{p}{p-k}$$

if k is even, and

$$b_k = \binom{p}{k} + \binom{p}{2}\binom{p-2}{k} + \binom{p}{4}\binom{p-4}{k} + ... + \binom{p}{p-k-1}\binom{k}{k-1}$$

if k odd. These are all obvious multiples of p. The case where k is negative is symmetrical. So since $(x + x^{-1} + 1)^{p-1}(x + x^{-1} + 1) = (x + x^{-1} + 1)^p$, through some algebraic manipulations one can see that $a_{i-1} + a_i + a_{i+1}$ is a multiple of p if $-p < i < p$ and $i \neq 0$. Also

$$a_{-1} + a_0 + a_1 = 1 \pmod p$$

,

$$a_{-p+1} + a_{-p+2} = 1 \pmod p,$$

and

$$a_p + a_{p-1} = 1 \pmod p.$$

So there are the following 3 cases: if $n = p \pmod 3$, $a_n = 0 \pmod p$; if $n = p - 1 \pmod 3$, $a_n = 1 \pmod p$; if $n = p + 1 \pmod 3$, $a_n = -1 \pmod p$. So, the answer is 4.

10. Find the number of ordered pairs of integers (m, n) with $0 \leq m, n \leq 22$ such that $k^2 + mk + n$ is not a multiple of 23 for all integers k.

Proposed by: Patrick Lu

Answer: $\boxed{253}$

Solution:

Equivalently we wish to find the number of monic irreducible polynomials in $\mathbb{Z}/23\mathbb{Z}$ of degree 2. Recall that for any prime p, $x^{p^n} - x$ is the product of all monic irreducible polynomials in $\mathbb{Z}/p\mathbb{Z}$ of degree dividing n. Then, if a is the number of monic irreducible polynomials in $\mathbb{Z}/23\mathbb{Z}$ of degree 1 and b is the number of monic irreducible polynomials in $\mathbb{Z}/23\mathbb{Z}$ of degree 2, $a + 2b = 23^2$. Since all degree 1 polynomials are irreducible, $a = 23$ and $b = \frac{23^2-23}{2} = 253$.

11. Suppose we have sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ and the function $f(x) = \frac{1}{x}$ such that for all n we have

- $a_{n+1} = f(f(a_n + b_n) - f(f(a_n) + f(b_n)))$
- $a_{n+2} = f(1 - a_n) - f(1 + a_n)$
- $b_{n+2} = f(1 - b_n) - f(1 + b_n)$

Given that $a_0 = \frac{1}{6}$ and $b_0 = \frac{1}{7}$, then $b_5 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find the sum of the prime factors of mn.

Proposed by: Vismay Sharan

Answer: $\boxed{1914}$

Solution: We see that if we take $\alpha_n = \tan^{-1}(a_n)$ and $\beta_n = \tan^{-1}(b_n)$ the first condition turns into the tangent addition formula and the second and third conditions turn into double angle formulas. So this means that if we only consider the angles the sequence for α is $\alpha_0, \alpha_0+\beta_0, 2\alpha_0, 2(\alpha_0+\beta_0), \ldots$, so then we can solve for β_1 and we get that $(\alpha_0+\beta_0)+\beta_1 = 2\alpha_0$ so $\beta_1 = \alpha_0 - \beta_0$ so we similarly get the sequence for β which is $\beta_0, \beta_0 - \alpha_0, 2\beta_0, 2(\alpha_0 - \beta_0)$ so this means that $b_5 = \tan(\beta_5) = \tan(4(\alpha_0 - \beta_0)) = \tan(4(\tan^{-1}(1/6) - \tan^{-1}(1/7)))$ which can be computed: $2^3 \cdot 3 \cdot 7 \cdot 11 \cdot 43/(881 \cdot 967)$, so $967 + 881 + 2 + 3 + 7 + 11 + 43 = 1914$.

12. Let ABC be a triangle with incenter I, circumcenter O, and A-excenter J_A. The incircle of $\triangle ABC$ touches side BC at a point D. Lines OI and J_AD meet at a point K. Line AK meets the circumcircle of $\triangle ABC$ again at a point $L \neq A$. If $BD = 11$, $CD = 5$, and $AO = 10$, the length of DL can be expressed as $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.

Proposed by: Patrick Lu

Answer: $\boxed{329}$

Solution: Full solution in progress