MMATHS 2023 Tiebreaker Round Solutions

Yale Math Competitions

October 2023

1. Let $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k} = \prod_{i=1}^{k} p_i^{e_i}$, where $p_1 < p_2 < ... < p_k$ are primes and $e_1, e_2, ..., e_k$ are positive integers, and let $f(n) = \prod_{i=1}^{k} e_i^{p_i}$. Find the number of integers n such that $2 \leq n \leq 2023$ and $f(n) = 128$.

   Proposed by: Benjamin Wu and Darwin Deng

   Answer: $\boxed{35}$

   Solution: We can see that $f(n)$ takes the prime factorization of n, and flips all of the exponents and bases. $128 = 2^7$, and we can see that the cases that would "flip" to 2^7 would be 7^2, $2^2 \times 5^2$, and $2^4 \times 3^2$. Also note that if we have p relatively prime to our cases and p has no exponents greater than 1 in it's prime factorization, multiplying our case by p and then flipping would result in $128 \times 1^n = 128$. We must then count how many valid p's there are for each case.

   For $49p$, we have $p = 1, 2, 3, 5, 6, 10, 11, 13, 15, 17, 19, 22, 23, 26, 29, 30, 31, 33, 34, 37, 38, 39, 41$ work, for a total of 23.

   For $100p$, we have $1, 3, 7, 11, 13, 17, 19$ work, for a total of 7.

   For $144p$, we have $p = 1, 5, 7, 11, 13$ work, for a total of 5.

   Then our answer is $23 + 7 + 5 = \boxed{35}$.

2. The lengths of the altitudes of $\triangle ABC$ are the roots of the polynomial $x^3 - 34x^2 + 360x - 1200$. Find the area of $\triangle ABC$.

   Proposed by: Benjamin Wu

   Answer: $\boxed{100}$

   Solution: Let a, b, and c be the sides of $\triangle ABC$, h_a, h_b, and h_c be the lengths of altitudes to a, b, and c, and K be the area of $\triangle ABC$. Then we have

   $$a = \frac{2K}{h_a}, b = \frac{2K}{h_b}, c = \frac{2K}{h_c}.$$

   We have $s = K(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c})$, and Heron's Formula gives $K = \sqrt{s(s-a)(s-b)(s-c)}$. If we let $h = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$, we can simplify the equation to

$$\begin{aligned} K &= \sqrt{K^4 \cdot h(h-2a)(h-2b)(h-2c)} \\ &= \frac{1}{\sqrt{h(h-2a)(h-2b)(h-2c)}} \\ &= \frac{1}{\sqrt{h \cdot (h^3 - 2h^2(a+b+c) + 4h(ab+bc+ac) - 8abc)}} \\ &= \frac{1}{\sqrt{h \cdot \left(h^3 - 2h^2\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) + 4h\left(\frac{1}{h_a h_b} + \frac{1}{h_b h_c} + \frac{1}{h_a h_c}\right) - \frac{8}{h_a h_b h_c}\right)}}. \end{aligned}$$

By Vieta's formulas,

$$h = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{h_a h_b + h_b h_c + h_c h_a}{h_a h_b h_c} = \frac{360}{1200} = \frac{3}{10}$$

$$\frac{1}{h_a h_b} + \frac{1}{h_b h_c} + \frac{1}{h_a h_c} = \frac{h_a + h_b + h_c}{h_a h_b h_c} = \frac{34}{1200}.$$

We then have that

$$\sqrt{h(h-2a)(h-2b)(h-2c)} = \sqrt{\frac{3}{10}\left(\frac{3}{10}\right)^3 - 2\left(\frac{3}{10}\right)^3 + 4 \cdot \frac{3}{10} \cdot \frac{34}{1200} - \frac{8}{1200}} = \frac{1}{100}$$

Then $K = \frac{1}{\frac{1}{100}} = \boxed{100}$.

Alternatively, while looking for roots, one can see that 10 is a root of the polynomial, and easily find the other two roots with the quadratic equation, which are $12 - 2\sqrt{6}$ and $12 + 2\sqrt{6}$. Plugging these into the equations found before can make the process a lot quicker.