MMATHS 2023 Individual Round Solutions

Yale Math Competitions

October 2023

1. Cat and Claire are having a conversation about Cat's favorite number. Cat says, "My favorite number is a two-digit multiple of 7."

   Claire asks, "If you just told me the tens digit of the number, would I know your number?"

   Cat says, "No. However, without knowing that, if I told you the tens digit of 100 minus my number, you could determine my favorite number."

   Claire says, "Now I know your favorite number!"

   What is Cat's favorite number?

   Proposed by: Jason Wang and Grant Zhang

   Answer: $\boxed{77}$

   Solution: Because just being told the tens digit of the number does not reveal the number, some other multiple of 7 must have the same tens digit as our number. However, from the second clue, there are no multiples of 7 which have the same tens digit after subtracting it from 100. Two numbers can share a tens digit but not share a tens digit after subtracting by 100 if and only if one of those numbers is a multiple of 10 (for any number $\overline{AB}$ with $B \neq 0$, 100 - $\overline{AB}$ has tens digit $10 - A - 1$, but for $\overline{A0}$, 100 - $\overline{A0}$ has tens digit $10 - A$). Then our number must be either 70 or 77; $100 - 77 = 23$ is the only number with tens digit 2 after subtraction, but $100 - 70$ has the same tens digit as $100 - 63 = 37$, so our number must be $\boxed{77}$.

2. In the Game of Life, each square in an infinite grid of squares is either shaded or blank. Every day, if a square shares an edge with exactly zero or four shaded squares, it becomes blank the next day. If a square shares an edge with exactly two or three squares, it becomes shaded the next day. Otherwise, it does not change. On day 1, each square is randomly shaded or blank with equal probability. If the probability that a given square is shaded on day 2 is $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a + b$.

   Proposed by: Stephen Xia

   Answer: $\boxed{7}$

   Solution: If our square started shaded, then we need it to have 1, 2, or 3 shaded neighbors to stay shaded. There are 16 total ways to shade these neighbors, and $\binom{4}{1} + \binom{4}{2} + \binom{4}{3} = 14$ of these ways are satisfactory. On the other hand, if our square started blank, then we need it to have exactly 2 or 3 shaded neighbors. There are $\binom{4}{2} + \binom{4}{3} = 10$ ways to do this. Since the square starts shaded or empty with equal probability, the total probability is $\frac{1}{2} \cdot \frac{14}{16} + \frac{1}{2} \cdot \frac{10}{16} = \frac{3}{4}$, so our answer is $3 + 4 = \boxed{7}$.

3. Simon expands factored polynomials with his favorite AI, ChatSFFT. However, he has not paid for a premium ChatSFFT account, so when he goes to expand $(m-a)(n-b)$, where a, b, m, n are integers, ChatSFFT returns the sum of the two factors instead of the product. However, when Simon plugs in certain pairs of integer values for m and n, he realizes that the value of ChatSFFT's result is the same as the real result. How many such pairs are there?

   Proposed by: MMATHS Problem Writers?

   Answer: $\boxed{2}$

   Solution: The mistaken expression is $(m-a)+(n-b)$, which we need to be equivalent to $(m-a)(n-b) = mn - an - bm + ab$. Solving, we have:

$$\begin{aligned} m + n - a - b &= mn - an - bm + ab \\ 0 &= mn - (a+1)n - (b+1)m + ab + a + b \\ 0 &= (m-(a+1))(n-(b+1)) - 1 \\ 1 &= (m-(a+1))(n-(b+1)) \end{aligned}$$

   Since a, b, m, n are all integers, we must have that both factors are 1 or -1. Therefore, the only solutions for (m, n) are $(a+2, b+2); (a, b)$, so the answer is $\boxed{2}$.

4. Let A and B be regular unit hexagons that share a center. Then, let $\mathcal{P}$ be the set of points contained in at least one of the hexagons. If the maximum possible area of $\mathcal{P}$ is X and the minimum possible area of $\mathcal{P}$ is Y, then the value of $X - Y$ can be expressed as $\frac{a\sqrt{b}-c}{d}$, where a, b, c, d are positive integers such that b is square-free and $\gcd(a, c, d) = 1$. Find $a+b+c+d$.

   Proposed by: Stephen Xia

   Answer: $\boxed{62}$

   Solution: The area of $\mathcal{P}$ is minimized when the overlap between the two hexagons is maximized; in other words, this is when A and B are the same, and thus is equivalent to the area of a singular unit hexagon. The area of $\mathcal{P}$ is maximized when the overlap between the two hexagons is minimized. This occurs when B is rotated by $30°$. In this case, the area of $\mathcal{P}$ can be expressed as the original hexagon's area with additional triangles portruding from each of the six sides. Thus, to find $X - Y$, it suffices to find the additional area added by these triangles. The height of each triangle can be expressed as the difference between center-to-corner and center-to-edge on a unit hexagon: $1 - \frac{\sqrt{3}}{2}$. We know the angle opposite from each height is $30°$ by rotation, so the base of each triangle is $2(\sqrt{3} - \frac{3}{2}) = 2\sqrt{3} - 3$. Thus, the total additional area from the six triangles can be expressed as $6(\frac{1}{2})(2\sqrt{3}-3)(1-\frac{\sqrt{3}}{2}) = \frac{21\sqrt{3}-36}{2}$. So, $a+b+c+d = 21+3+36+2 = \boxed{62}$

5. We call $\triangle ABC$ with centroid G *balanced* on side AB if the foot of the altitude from G onto line $\overline{AB}$ lies between A and B. $\triangle XYZ$, with $XY = 2023$ and $\angle ZXY = 120°$, is balanced on

XY. What is the maximum value of XZ?

Proposed by: Stephen Xia

Answer: $\boxed{4046}$

Solution: For easy visualization, for the rest of this solution let XY lie flat on the x-axis.

Let P be the midpoint of XZ. Then the centroid C of $\triangle XYZ$ will be located $2/3$ of the way from Y to P. Note that XZ is maximized when the centroid is directly "above" X, as increasing XZ any further will move the centroid past XY and make the triangle unbalanced.

Then because $YC = 2PC$, the horizontal distance from Y to C must be twice the horizontal distance from C to P. The horizontal distance from Y to C is just $XY = 2023$ (as we defined C to be directly above X). Then the horizontal distance from C to P must be $\frac{2023}{2}$. Because $\angle YXZ = 120°$, we have $\angle CXZ = 30°$. Then the horizontal distance from C to P is $\sin 30° \cdot XP = \frac{1}{2}XP$. Then $XP = 2023$, so $XZ = 2 \cdot 2023 = \boxed{4046}$.

6. Compute $\left|\sum_{i=1}^{2022}\sum_{j=1}^{2022}\cos\left(\frac{ij\pi}{2023}\right)\right|$.

Proposed by: Stephen Xia

Answer: $\boxed{1011}$

Solution: First consider some odd i. Then in our sum over all j, we can pair each j with $2023 - j$ to get:

$$\cos(\frac{ij\pi}{2023}) + \cos(\frac{i(2023-j)\pi}{2023})$$
$$\cos(\frac{ij\pi}{2023}) + \cos(i\pi - \frac{ij\pi}{2023})$$

When i is odd, this is equivalent to $\cos(x) + \cos(\pi - x) = 0$. Because we have an even number of j, we can create 1011 such pairs, each of which are equivalent to 0. So, $\sum_{j=1}^{2022}\cos\left(\frac{ij\pi}{2023}\right) = 0$ when i is odd.

Now consider $i = 2$. Then note that $\sum_{j=0}^{2022}\cos\left(\frac{2j\pi}{2023}\right)$ consists of 2023 roots of unity, so we must have $\sum_{j=0}^{2023}\cos\left(\frac{2j\pi}{2023}\right) = 0$. Then our target sum is simply the roots of unity without $j = 0$, so we have:

$$\sum_{j=1}^{2022}\cos\left(\frac{2j\pi}{2023}\right) = \sum_{j=0}^{2022}\cos\left(\frac{2j\pi}{2023}\right) - \cos(0) = -1$$

By symmetry, any even $i = 2k$ will also generate a sum such that $\sum_{j=0}^{2022}\cos\left(\frac{2kj\pi}{2023}\right) = 0$, so we have $\sum_{j=1}^{2022}\cos\left(\frac{2j\pi}{2023}\right) = -1$ for all even i.

There are 1011 even i, so we have:

$$\left|\sum_{i=1}^{2022}\sum_{j=1}^{2022}\cos\left(\frac{ij\pi}{2023}\right)\right| = |-1011| = \boxed{1011}$$

7. A 2023×2023 grid of lights begins with every light off. Each light is assigned a coordinate (x, y). For every distinct pair of lights (x_1, y_1), (x_2, y_2), with $x_1 < x_2$ and $y_1 > y_2$, all lights strictly between them (i.e. $x_1 < x < x_2$ and $y_2 < y < y_1$) are toggled. After this procedure is done, how many lights are on?

   Proposed by: Stephen Xia

   Answer: $\boxed{1022121}$

   Solution: We can consider each individual light— it suffices to count the number of times it is toggled. For any light (x, y), there are $(x-1)(2023-y)$ lights (x_1, y_1) with $x_1 < x, y_1 > y$, and $(2023-x)(y-1)$ lights (x_2, y_2) with $x_2 > x, y_2 < y$. Then every light (x, y) is toggled $(x-1)(2023-y)(2023-x)(y-1)$ times. A light is on if and only if it is toggled an odd number of times; then both x, y must be even. There are thus 1011^2 choices for coordinates that are both even, so $1011^2 = \boxed{1022121}$ lights are on.

8. Find the number of ordered pairs of integers (m, n) such that $0 \le m, n \le 2023$ and

$$m^2 \equiv \sum_{d|2023} n^d \pmod{2024}.$$

   Proposed by: Patrick Lu

   Answer: $\boxed{1012}$

   Solution: Define $f(n) = \sum_{d|2023} n^d$. By the Chinese Remainder Theorem, it suffices to find the number of solutions to $m^2 = f(n)$ in $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/11\mathbb{Z}$, and $\mathbb{Z}/23\mathbb{Z}$.

   We show that for any prime $p \equiv 3 \pmod 4$, there are exactly p solutions to $m^2 = f(n)$ in $\mathbb{Z}/p\mathbb{Z}$. For each $n \in \mathbb{Z}/p\mathbb{Z}$, we determine the number of $m \in \mathbb{Z}/p\mathbb{Z}$ such that $f(n) = m^2$. We use the fact that $f(-n) = f(n)$, as every term of $f(n)$ has odd degree. We consider two cases.

   (i) $f(n) \equiv 0 \pmod p$. Then $(n, 0)$ is the unique solution.

   (ii) $f(n) \not\equiv 0 \pmod p$. Note that for any $a \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$ where $p \equiv 3 \pmod 4$, exactly one of a and $-a$ is a quadratic residue. Then exactly one of $f(n)$ and $f(-n)$ is a quadratic residue. Combining the cases for n and $-n$, we see that there are 2 solutions in total.

   Thus there are 11 solutions in $\mathbb{Z}/11\mathbb{Z}$ and 23 solutions in $\mathbb{Z}/23\mathbb{Z}$. Now we find the number of solutions in $\mathbb{Z}/8\mathbb{Z}$. If $n \equiv 1 \pmod 2$, then $f(n) = 6n = m^2$ has no solutions, since $6n \equiv 2 \pmod 8$ and therefore cannot be a quadratic residue. If $n \equiv 0 \pmod 2$, then $f(n) = n = m^2$ has exactly four solutions: $(0, 0)$, $(0, 4)$, $(4, 2)$, and $(4, 6)$. Thus there are $4 \cdot 11 \cdot 23 = 1012$ solutions in total.

9. In $\triangle ABC$ with $\angle BAC = 60^\circ$, points D, E, and F lie on BC, AC, and AB, respectively, such that D is the midpoint of BC and $\triangle DEF$ is equilateral. If $BF = 1$ and $EC = 13$, then the area of $\triangle DEF$ can be written as $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers and b is not a perfect square. Compute $a + b + c$.

   Proposed by: Jeffrey Xu

   Answer: $\boxed{68}$

   Solution: Claim: Quadrilateral $BFEC$ is cyclic.
   Proof: For any point P on BC, let P_1 and P_2 be the corresponding points on segments AC and AB such that PP_1P_2 is equilateral. Suppose that there exists a point D' on BC such that quadrilateral $BD_2'D_1'C$ is cyclic. We will prove that D' and D must be the same point. We must have

$$\angle BD_2'D' = 120 - \angle AD_2'D_1' = \angle AD_1'D_2' = 180 - \angle D_2'D_1'C = \angle BD_2'D'$$

   so $BD' = D'D_2'$. Similarly, we have $CD' = D'D_1'$, so $BD' = D'D_2' = D'D_1' = CD'$, as $D'D_1'D_2'$ is equilateral. Thus, $BD' = CD'$, so $D' = D$, and our proof is complete.

   Because D is equidistant from B, F, E, and C, it must in fact be the circumcenter of quadrilateral $BFEC$. Thus, BE and CF are perpendicular to AC and AB, respectively. Then, we have $BF = AB - AF = 2AE - AF = 1$, and $CE = AC - AE = 2AF - AE = 13$. Solving gives $AE = 5$ and $AF = 9$, so $EF^2 = 5^2 + 9^2 - 2 \cdot 5 \cdot 9 \cdot \frac{1}{2} = 61$ by Law of Cosines. Finally, we have $[DEF] = \frac{EF^2\sqrt{3}}{4} = \frac{61\sqrt{3}}{4}$, so our answer is $61 + 3 + 4 = \boxed{68}$.

10. Consider the recurrence relation $x_{n+2} = 2x_{n+1} + x_n$, with $x_0 = 0, x_1 = 1$. What is the greatest common divisor of x_{2023} and x_{721}?

   Proposed by: Xifan Yu

   Answer: $\boxed{169}$

   Solution: We first claim that $x_{a+b} = x_a x_{b+1} + x_{a-1} x_b$ for all $a \geq 1, b \geq 0$. We can show this with induction.

   For $a = 1, b = 0$, this is trivial. Then given that this equality holds for $a, b \geq 0$, I show that it holds for $a, b + 1$. We have:

$$\begin{aligned}
x_{a+b+1} &= 2x_{a+b} + x_{a+b-1} \\
x_{a+b+1} &= 2(x_a x_{b+1} + x_{a-1} x_b) + x_a x_b + x_{a-1} x_{b-1} \\
x_{a+b+1} &= x_a(2x_{b+1} + x_b) + x_{a-1}(2x_b + x_{b-1}) \\
x_{a+b+1} &= x_a(2x_{b+1} + x_b) + x_{a-1}(2x_b + x_{b-1}) \\
x_{a+b+1} &= x_a x_{b+2} + x_{a-1} x_{b+1}
\end{aligned}$$

   So, this holds for $a, b+1$, and thus (because a, b are interchangeable) holds for all $a \geq 1, b \geq 0$.

   Now I show that x_a divides x_{ab} for any $a, b \geq 0$. For all $b = 0$, this is trivial. Now given that this holds for a, b, I show it holds for $a, b + 1$. We have:

$$x_{a(b+1)} = x_{ab+a} = x_{ab} x_{a+1} + x_{ab-1} x_a$$

x_a divides the second term clearly, and x_a divides x_{ab} in the first term by assumption, so x_a divides $x_{a(b+1)}$. Thus, this holds for all $b \geq 0$ for any chosen a, so it holds for all $a, b \geq 0$.

From here it can be seen that $\gcd(x_a, x_b) = x_{\gcd(a,b)}$; try using Euclidean algorithm. Then $\gcd(x_{2023}, x_{729}) = x_{\gcd(2023,729)} = x_7 = \boxed{169}$

11. A knight is on an infinite chessboard. After exactly 100 legal moves, how many different possible squares can it end on? A knight can move to any of the 8 closest squares not on the same row, column, or diagonal.

    Proposed by: Neil He

    Answer: $\boxed{70401}$

    Solution: The idea is to notice that the knight must end on a square of the same color, as we are taking an even number of steps (every jump changes the color of the square the knight is on). Then it suffices to find the outer boundaries of where the knight can reach. Viewing this as a graph with the knight starting at the origin, we can see a single move as changing one of x, y by 1 and 2 units, respectively (i.e. $x+1, y+2$). The knight can then reach any point with $x, y \leq 200$ and $x + y \leq 300$. This traces out an octagon, which is equivalent to a 401 by 401 square with four 100 by 100 triangles cut out from the corners. For each region, we divide by 2 as only half of the squares in that region are of the correct color (adding 1 in some cases, be careful with parity). Then overall, we get:

$$\frac{(401^2+1)}{2} - \frac{4 \cdot 100^2}{2 \cdot 2} = \boxed{70401}$$

12. Let ABC be a triangle with incenter I. The incircle ω of ABC is tangent to sides BC, CA, and AB at points D, E, and F, respectively. Let D' be the reflection of D over I. Let P be a point on ω such that $\angle ADP = 90^\circ$. $\mathcal{H}$ is a hyperbola passing through D', E, F, I, and P. Given that $\angle BAD = 45^\circ$ and $\angle CAD = 30^\circ$, the acute angle between the asymptotes of $\mathcal{H}$ can be expressed as $\left(\frac{m}{n}\right)^\circ$, where m and n are relatively prime positive integers. Find $m + n$.

    Proposed by: Patrick Lu, Ruben Carpenter

    Answer: $\boxed{167}$

    Solution: Full solution in progress