Indiv ID: ________

Team ID: ________

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12. ________

– 75 minutes
– no calculators
– positive integer answers

MMATHS **Name:** ____________________ **Team:** ____________________

2023 Individual Round

1. Cat and Claire are having a conversation about Cat's favorite number. Cat says, "My favorite number is a two-digit multiple of 7."

Claire asks, "If you just told me the tens digit of the number, would I know your number?"

Cat says, "No. However, without knowing that, if I told you the tens digit of 100 minus my number, you could determine my favorite number."

Claire says, "Now I know your favorite number!"

What is Cat's favorite number?

2. In the Game of Life, each square in an infinite grid of squares is either shaded or blank. Every day, if a square shares an edge with exactly zero or four shaded squares, it becomes blank the next day. If a square shares an edge with exactly two or three squares, it becomes shaded the next day. Otherwise, it does not change. On day 1, each square is randomly shaded or blank with equal probability. If the probability that a given square is shaded on day 2 is $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a+b$.

3. Simon expands factored polynomials with his favorite AI, ChatSFFT. However, he has not paid for a premium ChatSFFT account, so when he goes to expand $(m-a)(n-b)$, where a, b, m, n are integers, ChatSFFT returns the sum of the two factors instead of the product. However, when Simon plugs in certain pairs of integer values for m and n, he realizes that the value of ChatSFFT's result is the same as the real result in terms of a and b. How many such pairs are there?

4. Let A and B be regular unit hexagons that share a center. Then, let $\mathcal{P}$ be the set of points contained in at least one of the hexagons. If the maximum possible area of $\mathcal{P}$ is X and the minimum possible area of $\mathcal{P}$ is Y, then the value of $X-Y$ can be expressed as $\frac{a\sqrt{b}-c}{d}$, where a, b, c, d are positive integers such that b is square-free and $\gcd(a, c, d)=1$. Find $a+b+c+d$.

5. We call $\triangle ABC$ with centroid G *balanced* on side AB if the foot of the altitude from G onto line $\overline{AB}$ lies between A and B. $\triangle XYZ$, with $XY = 2023$ and $\angle ZXY = 120^\circ$, is balanced on XY. What is the maximum value of XZ?

6. Compute $\left|\sum_{i=1}^{2022}\sum_{j=1}^{2022}\cos\left(\frac{ij\pi}{2023}\right)\right|$.

7. A 2023×2023 grid of lights begins with every light off. Each light is assigned a coordinate (x, y). For every distinct pair of lights (x_1, y_1), (x_2, y_2), with $x_1 < x_2$ and $y_1 > y_2$, all lights strictly between them (i.e. $x_1 < x < x_2$ and $y_2 < y < y_1$) are toggled. After this procedure is done, how many lights are on?

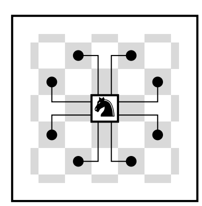
8. Find the number of ordered pairs of integers (m, n) such that $0 \le m, n \le 2023$ and

$$m^2 \equiv \sum_{d \mid 2023} n^d \pmod{2024}.$$

9. In $\triangle ABC$ with $\angle BAC = 60^\circ$, points D, E, and F lie on BC, AC, and AB, respectively, such that D is the midpoint of BC and $\triangle DEF$ is equilateral. If $BF = 1$ and $EC = 13$, then the area of $\triangle DEF$ can be written as $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers and b is not a perfect square. Compute $a+b+c$.

10. Consider the recurrence relation $x_{n+2} = 2x_{n+1} + x_n$, with $x_0 = 0, x_1 = 1$. What is the greatest common divisor of x_{2023} and x_{721}?

11. A knight is on an infinite chessboard. After exactly 100 legal moves, how many different possible squares can it end on? A knight can move to any of the 8 closest squares not on the same row, column, or diagonal, as illustrated in the figure to the right.



12. Let ABC be a triangle with incenter I. The incircle ω of ABC is tangent to sides BC, CA, and AB at points D, E, and F, respectively. Let D' be the reflection of D over I. Let P be a point on ω such that $\angle ADP = 90^\circ$. $\mathcal{H}$ is a hyperbola passing through D', E, F, I, and P. Given that $\angle BAD = 45^\circ$ and $\angle CAD = 30^\circ$, the acute angle between the asymptotes of $\mathcal{H}$ can be expressed as $\left(\frac{m}{n}\right)^\circ$, where m and n are relatively prime positive integers. Find $m+n$.