MMATHS Team: ____________ Team ID: ________

2023 Team Round

1. Lucy has 8 children, each of whom has a distinct favorite integer from 1 to 10, inclusive. The smallest number that is a perfect multiple of all of these favorite numbers is 1260, and the average of these favorite numbers is at most 5. Find the sum of the four largest numbers.

2. 20 players enter a chess tournament in which each player will play every other player exactly once. Some competitors are cheaters and will cheat in every game they play, but the rest of the competitors are not cheaters. A game is *cheating* if both players cheat, and a game is *half-cheating* if one player cheats and one player does not. If there were 68 more half-cheating games than cheating games, how many of the players are cheaters?

3. There are 360 permutations of the letters in $MMATHS$. When ordered alphabetically, starting from $AHMMST$, $MMATHS$ is the nth permutation. What is n?

4. How many distinct real numbers x satisfy the equation $4\cos^3(x) + \sqrt{x} = 3\sin(x) + \cos(3x)$?

5. $\omega_A, \omega_B, \omega_C$ are three concentric circles with radii 2, 3, and 7, respectively. We say that a point P in the plane is *nice* if there are points A, B, and C on ω_A, ω_B, and ω_C, respectively, such that P is the centroid of $\triangle ABC$. If the area of the smallest region of the plane containing all nice points can be expressed as $\frac{a\pi}{b}$, where a and b are relatively prime positive integers, what is $a+b$?

6. 10 points are drawn on each of two parallel lines. What is the largest number of acute triangles of positive area that can be formed using three of these 20 points as vertices?

7. $ABCD$ is a regular tetrahedron of side length 4. Four congruent spheres are inside $ABCD$ such that each sphere is tangent to exactly three of the faces, the spheres have distinct centers, and the four spheres are concurrent at one point. Let v be the volume of one of the spheres. If v^2 can be written as $\frac{a}{b}\pi^2$, where a and b are relatively prime positive integers, find $a+b$.

8. 30 people sit around a table, some of which are Yale students. Each person is asked if the person to their right is a Yale student. Yale students will always answer correctly, but non-Yale students will answer randomly. Find the smallest possible number of Yale students such that, after hearing everyone's answers and knowing the number of Yale students, it is possible to identify *for certain* at least one Yale student.

9. Let $(x + x^{-1} + 1)^{40} = \sum_{i=-40}^{40} a_i x^i$. Find the remainder when $\sum_{p \text{ prime}} a_p$ is divided by 41.

10. Find the number of ordered pairs of integers (m, n) with $0 \le m, n \le 22$ such that $k^2 + mk + n$ is not a multiple of 23 for all integers k.

11. Suppose we have sequences $(a_n)_{n\ge 0}$ and $(b_n)_{n\ge 0}$ and the function $f(x) = \frac{1}{x}$ such that for all n we have

- $a_{n+1} = f(f(a_n + b_n) - f(f(a_n) + f(b_n)))$
- $a_{n+2} = f(1 - a_n) - f(1 + a_n)$
- $b_{n+2} = f(1 - b_n) - f(1 + b_n)$

Given that $a_0 = \frac{1}{6}$ and $b_0 = \frac{1}{7}$, then $b_5 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find the sum of the prime factors of mn.

12. Let ABC be a triangle with incenter I, circumcenter O, and A-excenter J_A. The incircle of $\triangle ABC$ touches side BC at a point D. Lines OI and J_AD meet at a point K. Line AK meets the circumcircle of $\triangle ABC$ again at a point $L \neq A$. If $BD = 11$, $CD = 5$, and $AO = 10$, the length of DL can be expressed as $\frac{m\sqrt{p}}{n}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________
7. ________
8. ________
9. ________
10. ________
11. ________
12. ________

– 45 minutes
– no calculators
– positive integer answers