Math Majors of America Tournament for High Schools

2020 Mixer Test

Name: __________

Team : __________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________
7. __________
8. __________
9. __________
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11. __________
12. __________
13. __________
14. __________
15. __________

– 75 min, no calculators, integer answers

1. There are five boys and five girls in a class. Suppose that they pair off in the following manner: two boys pair, two girls pair, and the rest of the class divides into boy-girl pairs. How many ways are there for this to occur?

2. Annie and Britta are playing a game. Annie picks a triple of single-digit positive integers and gives Britta the following hints: 1) Her numbers are all distinct; 2) The sum of her numbers is greater than or equal to 20; 3) The product of her numbers is divisible by 32. Britta computes the sum of all single-digit positive integers that Annie didn't choose. What number did Britta compute?

3. 8 ants, uncreatively denoted by $\text{Ant}_1, \text{Ant}_2, \cdots, \text{Ant}_8$, must each stand at a vertex of a regular octagon such that each ant stands at exactly one vertex, and the line segments joining Ant_i and Ant_{i+1} for $i = 1, 2, 3, \cdots, 7$ do not intersect except at the vertices of the octagon. How many ways can they do this? (Rotations and reflections *are* considered distinct.)

4. Claire and Cat are at a party with 10 other people. Every 10 minutes, two random people get up and leave, and this continues for an hour (until everybody has left). If the probability that Claire leaves before Cat can be expressed as $\frac{a}{b}$ in simplest terms, find $a + b$.

5. Given that integers a and b with $a + b \neq 0$ satisfy the equation $a + b - 2 = \frac{a+1}{b} + \frac{b+1}{a}$, find the sum of all possible values of a.

6. Suppose that Ω is a circle with center O, and let A, B, and C be points on Ω satisfying $AB = AC$. Suppose that there exists a circle ω centered at O tangent to $\overline{AB}$, $\overline{AC}$, and the arc $\widehat{BC}$ of the circle centered at A passing through B and C. If the ratio of the area of ω to the area of Ω can be expressed as $\frac{a}{b}$ for integers a, b satisfying $\gcd(a, b) = 1$, then find $a + b$.

7. Jennifer rolls three fair six-sided dice, with integer labels from 1 to 6. Let N be the product of the numbers she rolls; i.e., if she rolls $1, 4,$ and 5, then $N = 20$. If the probability that N has exactly 8 factors can be expressed as $\frac{a}{b}$ in simplest terms, find $a + b$.

8. How many ways are there to arrange four rooks on an four-by-four chessboard such that exactly two pairs of rooks attack each other? (Rooks are said to be attacking one another if they lie in the same row or column, without any rooks in between them.)

9. Let ABC be an equilateral triangle, and consider points D, E on line BC such that $BD = CE = \frac{BC}{3}$ and $DE = BC$. The value of $\sin \angle DAE$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers, b is squarefree, and $\gcd(a, c) = 1$. Find $a + b + c$.

10. Given an integer n, $f(n)$ returns the number of distinct positive integers a such that $\frac{na}{n+a}$ is an integer. If $A = f(3) + f(6) + f(12) + \cdots + f(3 \cdot 2^k) + \cdots + f(3 \cdot 2^{673})$, find the remainder when A is divided by 2020.

11. Let a, b be two real numbers such that

$$\sqrt[3]{a} - \sqrt[3]{b} = 10, \qquad ab = \left(\frac{8 - a - b}{6}\right)^3.$$

Find $a - b$.

12. Consider the function $f(n) = n^2 + n + 1$. For each n, let d_n be the smallest positive integer with $\gcd(n, d_n) = 1$ and $f(n) \mid f(d_n)$. Find $d_6 + d_7 + d_8 + d_9 + d_{10}$.

13. Suppose that ABC is a triangle with $AB = 13, BC = 14,$ and $CA = 15$; let M and N be the midpoints of AB and AC, and suppose that D is the foot of the A-altitude. Point X is chosen on the circumcircle of triangle DMN such that if Y is the reflection of X in MN, the length AY is maximal among all possible choices of X. If $AX = \frac{\sqrt{a}}{b}$ for integers a and b with a squarefree, find $a + b$.

14. Let x, y be positive reals such that $x \neq y$. Find the minimum possible value of $(x + y)^2 + \frac{54}{xy(x-y)^2}$.

15. Define the function $f(n)$ for positive integers n as follows: if n is prime, then $f(n) = 1$; and $f(ab) = a \cdot f(b) + f(a) \cdot b$ for all positive integers a and b. How many positive integers n less than 5^{50} have the property that $f(n) = n$?