Mathathon Round 1 (4 points each)

1. Let n be a two-digit positive integer. What is the maximum possible sum of the prime factors of $n^2 - 25$?

2. Angela has ten numbers $a_1, a_2, a_3, \cdots, a_{10}$. She wants them to be a permutation of the numbers $\{1, 2, 3, \cdots, 10\}$ such that for each $1 \leq i \leq 5$, $a_i \leq 2i$, and for each $6 \leq i \leq 10$, $a_i \leq 2i - 10$. How many ways can Angela choose a_1 through a_{10}?

3. Find the number of three-by-three grids such that

- the sum of the entries in each row, column, and diagonal passing through the center square is the same, and
- the entries in the nine squares are the integers between 1 and 9 inclusive, each integer appearing in exactly one square.

Mathathon Round 2 (5 points each)

4. Suppose that $P(x)$ is a quadratic polynomial such that the sum and product of its two roots are equal to each other. There is a real number a that $P(1)$ can never be equal to. Find a.

5. Find the number of ordered pairs (x, y) of positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{k}$ and k is a factor of 60.

6. Let ABC be a triangle with $AB = 5, AC = 4$, and $BC = 3$. With $B = B_0$ and $C = C_0$, define the infinite sequences of points $\{B_i\}$ and $\{C_i\}$ as follows: for all $i \geq 1$, let B_i be the foot of the perpendicular from C_{i-1} to AB, and let C_i be the foot of the perpendicular from B_i to AC. Find $C_0C_1(AC_0 + AC_1 + AC_2 + AC_3 + \cdots)$.

Mathathon Round 3 (6 points each)

7. If $\ell_1, \ell_2, \ldots, \ell_{10}$ are distinct lines in the plane and $p_1, \ldots, p_{100}$ are distinct points in the plane, then what is the maximum possible number of ordered pairs (ℓ_i, p_j) such that p_j lies on ℓ_i?

8. Before Andres goes to school each day, he has to put on a shirt, a jacket, pants, socks, and shoes. He can put these clothes on in any order obeying the following restrictions: socks come before shoes, and the shirt comes before the jacket. How many distinct orders are there for Andres to put his clothes on?

9. There are ten towns, numbered 1 through 10, and each pair of towns is connected by a road. Define a backwards move to be taking a road from some town a to another town b such that $a > b$, and define a forwards move to be taking a road from some town a to another town b such that $a < b$. How many distinct paths can Ali take from town 1 to town 10 under the conditions that

- she takes exactly one backwards move and the rest of her moves are forward moves, and
- the only time she visits town 10 is at the very end?

One possible path is $1 \to 3 \to 8 \to 6 \to 7 \to 8 \to 10$.

Mathathon Round 4 (7 points each)

10. How many prime numbers p less than 100 have the properties that $p^5 - 1$ is divisible by 6 and $p^6 - 1$ is divisible by 5?

11. Call a four-digit integer $\overline{d_1d_2d_3d_4}$ *primed* if 1) d_1, d_2, d_3, and d_4 are all prime numbers, and 2) the two-digit numbers $\overline{d_1d_2}$ and $\overline{d_3d_4}$ are both prime numbers. Find the sum of all primed integers.

12. Suppose that ABC is an isosceles triangle with $AB = AC$, and suppose that D and E lie on $\overline{AB}$ and $\overline{AC}$, respectively, with $\overline{DE} \parallel \overline{BC}$. Let r be the length of the inradius of triangle ADE. Suppose that it is possible to construct two circles of radius r, each tangent to one another and internally tangent to three sides of the trapezoid $BDEC$. If $\frac{BC}{r} = a + \sqrt{b}$ for positive integers a and b with b squarefree, then find $a + b$.

Mathathon Round 5 (8 points each)

13. A palindrome is a number that reads the same forward as backwards; for example, 121 and 36463 are palindromes. Suppose that N is the maximal possible difference between two consecutive three-digit palindromes. Find the number of pairs of consecutive palindromes (A, B) satisfying $A < B$ and $B - A = N$.

14. Suppose that x, y, and z are complex numbers satisfying $x + \frac{1}{yz} = 5$, $y + \frac{1}{zx} = 8$, and $z + \frac{1}{xy} = 6$. Find the sum of all possible values of xyz.

15. Let Ω be a circle with radius $25\sqrt{2}$ centered at O, and let C and J be points on Ω such that the circle with diameter $\overline{CJ}$ passes through O. Let Q be a point on the circle with diameter $\overline{CJ}$ satisfying $OQ = 5\sqrt{2}$. If the area of the region bounded by $\overline{CQ}, \overline{QJ}$, and minor arc $\widehat{JC}$ on Ω can be expressed as $\frac{a\pi - b}{c}$ for integers a, b, and c with $\gcd(a, c) = 1$, then find $a + b + c$.

Mathathon Round 6 (9 points each)

16. Veronica writes N integers between 2 and 2020 (inclusive) on a blackboard, and she notices that no number on the board is an integer power of another number on the board. What is the largest possible value of N?

17. Let ABC be a triangle with $AB = 12$, $AC = 16$, and $BC = 20$. Let D be a point on AC, and suppose that I and J are the incenters of triangles ABD and CBD, respectively. Suppose that $DI = DJ$. Find IJ^2.

18. For each positive integer a, let $P_a = \{2a, 3a, 5a, 7a, \ldots\}$ be the set of all prime multiples of a. Let $f(m, n) = 1$ if P_m and P_n have elements in common, and let $f(m, n) = 0$ if P_m and P_n have no elements in common. Compute

$$\sum_{1 \le i < j \le 50} f(i, j)$$

(i.e. compute $f(1,2) + f(1,3) + \cdots + f(1,50) + f(2,3) + f(2,4) + \cdots + f(49,50)$.)

Mathathon Round 7 (10 points each)

19. How many ways are there to put the six letters in "MMATHS" in a two-by-three grid such that the two "M"s do not occupy adjacent squares and such that the letter "A" is not directly above the letter "T" in the grid? (Squares are said to be adjacent if they share a side.)

20. Luke is shooting basketballs into a hoop. He makes any given shot with fixed probability p with $p < 1$, and he shoots n shots in total with $n \ge 2$. Miraculously, in n shots, the probability that Luke makes exactly two shots in is twice the probability that Luke makes exactly one shot in! If p can be expressed as $\frac{k}{100}$ for some integer k (not necessarily in lowest terms), find the sum of all possible values for k.

21. Let $ABCD$ be a rectangle with $AB = 24$ and $BC = 72$. Call a point P *goofy* if it satisfies the following conditions:

- P lies within $ABCD$;
- for some points F and G lying on sides $\overline{BC}$ and $\overline{DA}$ such that the circles with diameter $\overline{BF}$ and $\overline{DG}$ are tangent to one another, P lies on their common internal tangent.

Find the smallest possible area of a polygon that contains every single goofy point inside it.