Name: __________

Team : __________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

7. __________

8. __________

9. __________

10. __________

11. __________

12. __________

– 75 minutes
– no calculators
– all answers are integers

Math Majors of America Tournament for High Schools

2020 Individual Test

1. A nine-digit number has the form $\overline{6ABCDEFG3}$, where every three consecutive digits sums to 13. Find D.

2. Let b and c be real numbers not both equal to 1 such that $1, b, c$ is an arithmetic progression and $1, c, b$ is a geometric progression. What is $100(b - c)$?

3. Suppose that three prime numbers p, q, and r satisfy the equations $pq + qr + rp = 191$ and $p + q = r - 1$. Find $p + q + r$.

4. Let $ABCD$ be a square of side length 4. Points E and F are chosen on sides BC and DA, respectively, such that $EF = 5$. Find the sum of the minimum and maximum possible areas of trapezoid $BEDF$.

5. For some positive integers $m > n$, the quantities $a = \text{lcm}(m, n)$ and $b = \gcd(m, n)$ satisfy $a = 30b$. If $m - n$ divides a, then what is the value of $\frac{m+n}{b}$?

6. Prair has a box with some combination of red and green balls. If she randomly draws two balls out of the box (without replacement), the probability of drawing two balls of the same color is equal to the probability of drawing two balls of different colors! How many possible values between 200 and 1000 are there for the total number of balls in the box?

7. Suppose that ABC is a triangle with $AB = 6, BC = 12$, and $\angle B = 90°$. Point D lies on side BC, and point E is constructed on AC such that $\angle ADE = 90°$. Given that $DE = EC = \frac{a\sqrt{b}}{c}$ for positive integers a, b, and c with b squarefree and $\gcd(a, c) = 1$, find $a + b + c$.

8. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences such that $a_ib_i - a_i - b_i = 0$ and $a_{i+1} = \frac{2-a_ib_i}{1-b_i}$ for all $i \geq 1$. If $a_1 = 1 + \frac{1}{\sqrt[4]{2}}$, then what is b_6?

9. In how many ways can Irena can write the integers 1 through 7 in a line such that whenever she looks at any three consecutive (in the line) numbers, the largest is not in the rightmost position?

10. Let $f(x)$ be a quadratic polynomial such that $f(f(1)) = f(-f(-1)) = 0$ and $f(1) \neq -f(-1)$. Suppose furthermore that the quadratic $2f(x)$ has coefficients that are nonzero integers. Find $f(0)$.

11. Let triangle $\triangle ABC$ have side lengths $AB = 7, BC = 8$, and $CA = 9$, and let M and D be the midpoint of $\overline{BC}$ and the foot of the altitude from A to $\overline{BC}$, respectively. Let E and F lie on $\overline{AB}$ and $\overline{AC}$, respectively, such that $m\angle AEM = m\angle AFM = 90°$. Let P be the intersection of the angle bisectors of $\angle AED$ and $\angle AFD$. If MP can be written as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, and c with b squarefree and $\gcd(a, c) = 1$, then find $a + b + c$.

12. Let $p(x)$ be the monic cubic polynomial with roots $\sin^2(1°)$, $\sin^2(3°)$, and $\sin^2(9°)$. Suppose that $p\left(\frac{1}{4}\right) = \frac{\sin(a°)}{n\sin(b°)}$, where $0 < a, b \leq 90$ and a, b, n are positive integers. What is $a + b + n$?