Math Majors of America Tournament for High Schools

2014 Mixer Test

Team : __________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

7. __________

8. __________

9. __________

10. write on back

11. write on back

– 75 minutes
– no calculators
– simplify answers

1. How many real roots does the equation $2x^7 + x^5 + 4x^3 + x + 2 = 0$ have?

2. Given that $f(n) = 1 + \sum_{j=1}^{n}(1 + \sum_{i=1}^{j}(2i+1))$, find the value of $f(99) - \sum_{i=1}^{99} i^2$.

3. A rectangular prism with dimensions $1 \times a \times b$, where $1 < a < b < 2$, is bisected by a plane bisecting the longest edges of the prism. One of the smaller prisms is bisected in the same way. If all three resulting prisms are similar to each other and to the original box, compute ab. Note: Two rectangular prisms of dimensions $p \times q \times r$ and $x \times y \times z$ are similar if $\frac{p}{x} = \frac{q}{y} = \frac{r}{z}$.

4. For fixed real values of p, q, r and s, the polynomial $x^4 + px^3 + qx^2 + rx + s$ has four non real roots. The sum of two of these roots is $4 + 7i$, and the product of the other two roots is $3 - 4i$. Compute q.

5. There are 10 seats in a row in a theater. Say we have an infinite supply of indistinguishable good kids and bad kids. How many ways can we seat 10 kids such that no two bad kids are allowed to sit next to each other?

6. There are an infinite number of people playing a game. They each pick a different positive integer k, and they each win the amount they chose with probability $\frac{1}{k^3}$. What is the expected amount that all of the people win in total?

7. There are 100 donuts to be split among 4 teams. Your team gets to propose a solution about how the donuts are divided amongst the teams. (Donuts may not be split.) After seeing the proposition, every team either votes in favor or against the propisition. The proposition is adopted with a majority vote or a tie. If the proposition is rejected, your team is eliminated and will never receive any donuts. Another remaining team is chosen at random to make a proposition, and the process is repeated until a proposition is adopted, or only one team is left. No promises or deals need to be kept among teams besides official propositions and votes. Given that all teams play optimally to maximize the expected value of the number of donuts they receive, are completely indifferent as to the success of the other teams, but they would rather not eliminate a team than eliminate one (if the number of donuts they receive is the same either way), then how much should your team propose to keep?

8. Dominic, Mitchell, and Sitharthan are having an argument. Each of them is either credible or not credible – if they are credible then they are telling the truth. Otherwise, it is not known whether they are telling the truth. At least one of Dominic, Mitchell, and Sitharthan is credible. Tim knows whether Dominic is credible, and Ethan knows whether Sitharthan is credible. The following conversation occurs, and Tim and Ethan overhear:

> Dominic: "Sitharthan is not credible."
> Mitchell: "Dominic is not credible."
> Sitharthan: "At least one of Dominic or Mitchell is credible."

Then, at the same time, Tim and Ethan both simultaneously exclaim: "I can't tell exactly who is credible!" They each then think for a moment, and they realize that they can. If Tim and Ethan always tell the truth, then write on your answer sheet exactly which of the other three are credible.

9. Pick an integer n between 1 and 10. If no other team picks the same number, we'll give you $\frac{n}{10}$ points.

10. Many quantities in high-school mathematics are left undefined. Propose a definition or value for the following expressions and justify your response for each. We'll give you $\frac{1}{5}$ points for each reasonable argument.

(i) $(.5)!$ (ii) $\infty \cdot 0$ (iii) 0^0 (iv) $\prod_{x \in \emptyset} x$ (v) $1 - 1 + 1 - 1 + \cdots$

11. On the back of your answer sheet, write the "coolest" math question you know, and include the solution. If the graders like your question the most, then you'll get a point. (With your permission, we might include your question on the Mixer next year!)