1. Suppose $\frac{x}{y} = 0.\overline{ab}$ where $x$ and $y$ are relatively prime positive integers and $ab + a + b + 1$ is a multiple of 12. Find the sum of all possible values of $y$.

**Answer:** 152

**Solution:** $ab+a+b+1 = (a+1)(b+1)$. Therefore $a+1$ is either 2, 3, 4, or 6, and $b+1$ is in $\{6\}$, $\{4,8\}$, $\{3,6,9\}$, or $\{2,4,6,8\}$, respectively. Note that $0.\overline{ab} = (10a+b)\frac{\frac{1}{100}}{1-\frac{1}{100}} = \frac{10a+b}{99}$. So we have 10 possibilities for $\frac{x}{y}$ : $\frac{15}{99} = \frac{5}{33}, \frac{23}{99}, \frac{27}{99} = \frac{3}{11}, \frac{32}{99}, \frac{35}{99}, \frac{38}{99}, \frac{51}{99} = \frac{17}{33}, \frac{53}{99}, \frac{55}{99}, \frac{57}{99} = \frac{19}{33}$. This gives us values of 33, 99, 11, and 9 for $y$, which sum to $33 + 99 + 11 + 9 = 152$.

2. Let $A$ be the set of points $\{(0,0), (2,0), (0,2), (2,2), (3,1), (1,3)\}$. How many distinct circles pass through at least three points in $A$?

**Answer:** 13

**Solution:** There are $\binom{6}{3} = 20$ ways to choose subsets of 3 points, each of which defines a circle except for $\{(1,3), (2,2), (3,1)\}$ (which are collinear). Each of the 2 rectangles is counted 4 times, so the answer is $20 - 1 - 2(3) = 13$.

3. Jack and Jill need to bring pails of water home. The river is the $x$-axis, Jack is initially at the point $(-5,3)$, Jill is initially at the point $(6,1)$, and their home is at the point $(0,h)$ where $h > 0$. If they take the shortest paths home given that each of them must make a stop at the river, they walk exactly the same total distance. What is $h$?

**Answer:** $\frac{3}{4}$

**Solution:** Consider reflections across the $x-$axis. The total distance traveled is the total distance to the point $(0,-h)$. So the shortest paths are straight lines and we get $5^2 + (h+3)^2 = 6^2 + (h+1)^2 \to 34 + 6h + h^2 = 37 + 2h + h^2 \to 4h = 3 \to h = \frac{3}{4}$.

4. What is the largest perfect square which is not a multiple of 10 and which remains a perfect square if the ones and tens digits are replaced with zeroes?

**Answer:** 1681

**Solution:** Let $x^2 = 100y^2 + z$ be a perfect square, for $0 < z \leq 99$. Therefore, $0 < x^2 - 100y^2 \leq 99 \to 0 < (x+10y)(x-10y) \leq 99$. Then $(x-10y) > 0$ and $(x+10y) \leq 99 \to x \leq 49$. From here it is quick to check that $x = 41$ is the largest solution, and $41^2 = 1681$.

5. In convex polygon $\mathcal{P}$, each internal angle measure (in degrees) is a distinct integer. What is the maximum possible number of sides $\mathcal{P}$ could have?

**Answer:** 26

**Solution:** Suppose $\mathcal{P}$ has $n$ sides. In order for this to be possible, we must have $(180-1) + (180-2) + \cdots + (180-n) \geq 180(n-2)$ i.e. $180n - \frac{(n)(n+1)}{2} \geq 180n - 360 \to (n)(n+1) \leq 720$. By trial and error, $n = 26$ is the largest integer that works.

6. How many polynomials $p(x)$ of degree exactly 3 with real coefficients satisfy

$$p(0), p(1), p(2), p(3) \in \{0, 1, 2\}?$$

**Answer:** 72

**Solution:** Any set of $n$ points can be interpolated by a polynomial of degree at most $(n-1)$. So for each of the $3^4 = 81$ possible assignments of $p(0), p(1), p(2), p(3)$, there exists an interpolating polynomial of degree at most 3. It remains to subtract off the number of assignments that can be interpolated by a polynomial of degree at most 2. Note that the assignments $p(1) = a, p(2) = b, p(3) = c$ define a unique interpolating polynomial of degree at most 2, then it remains only to check if $p(0) \in \{0, 1, 2\}$. Write $p(x) = dx^2 + ex + f$.

$x = 1$ gives $a = d+e+f$, $x = 2$ gives $b = 4d+2e+f$, and $x = 3$ gives $c = 9d+3e+f$. Then $2a-b = -2d+f$ and $3a-c = -6d+2f$. And $3(2a-b)-(3a-c) = 3f-2f \longrightarrow f = 3(a-b)+c$. Now do case work. Case 1: $c = 0$. Then we need $3(a-b)+0 \in \{0,1,2\} \longrightarrow a = b$ yields 3 possibilities. Case 2: $c = 1$. Then $3(a-b)+1 \in \{0,1,2\} \longrightarrow a = b$ again yields 3 possibilities. Case 3: $c = 2$. Then $3(a-b)+2 \in \{0,1,2\} \longrightarrow a = b$ again yields 3 possibilities. So our final answer is $81-(3+3+3) = 72$. Note that the quadratics may also be counted using symmetry arguments.

7. Six spheres, each with radius $4$, are resting on the ground. Their centers form a regular hexagon, and adjacent spheres are tangent. A seventh sphere, with radius $13$, rests on top of and is tangent to all six of these spheres. How high above the ground is the center of the seventh sphere?

**Answer:** $19$

**Solution:** Centers of opposite spheres on the ground are $16$ apart, and centers of the small spheres are $4+13 = 17$ away from the center of the large sphere. This gives us an isosceles triangle with sides $16$, $17$, and $17$. We can break this into two $8$, $15$, $17$ right triangles. Therefore the center of the large sphere is $15$ above the centers of the small spheres, which are $4$ above the ground.

8. You have a paper square. You may fold it along any line of symmetry. (That is, the layers of paper must line up perfectly.) You then repeat this process using the folded piece of paper. If the direction of the folds does not matter, how many ways can you make exactly eight folds while following these rules?

**Answer:** $314$

**Solution:** Once you get a triangle, you're stuck with it. You can get to a square after $0$, $2$, $4$, or $6$ cuts, then you have $2$ options for making it into a triangle, then there are no more choices. This is $2(\binom{0}{0}+\binom{2}{1}+\binom{4}{2}+\binom{6}{3}) = 2(1+2+6+20) = 58$. If it never ends up as a triangle, there are $2$ options at each cut to keep it a rectangle, hence we get another $2^8 = 256$ options. The total is $58+256 = 314$.

9. Quadrilateral $ABCD$ has $\overline{AB} = 40$, $\overline{CD} = 10$, $\overline{AD} = \overline{BC}$, $m\angle BAD = 20°$, and $m\angle ABC = 70°$. What is the area of quadrilateral $ABCD$?

**Answer:** $375$

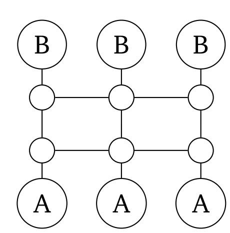
**Solution:** Place $4$ identical copies of $ABCD$ in a square of side length $40$; it leaves a hole in the middle which is a square of side length $10$. So the area of $ABCD$ is $\frac{1}{4}(40^2-10^2) = \frac{1500}{4} = 375$.

10. We say that a permutation $\sigma$ of the set $\{1,2,\ldots n\}$ *preserves divisibilty* if $\sigma(a)$ divides $\sigma(b)$ whenever $a$ divides $b$. How many permutations of $\{1,2,...,40\}$ preserve divisibility? (A permutation of $\{1,2,\ldots n\}$ is a function $\sigma$ from $\{1,2,\ldots n\}$ to itself such that for any $b \in \{1,2,\ldots n\}$, there exists some $a \in \{1,2,\ldots n\}$ satisfying $\sigma(a) = b$.)

**Answer:** $96$

**Solution:** Note that $\sigma$ is completely determined by where it sends the primes because then prime factorization lets us compute $\sigma(m)$ for composite $m$ (and it's clear that $\sigma(1) = 1$ because it needs to divide everything). The primes less than $40$ are $2$, $3$, $5$, $7$, $11$, $13$, $17$, $19$, $23$, $29$, $31$, and $37$. These primes divide $20$, $13$, $8$, $5$, $3$, $3$, $2$, $2$, $1$, $1$, $1$, $1$ integers in the range $1-40$, respectively. It is not difficult to see that each prime must be mapped to a prime which divides the same number of integers in the range as itself, so we can permute the elements of the sets $\{11,13\}$, $\{17,19\}$, and $\{23,29,31,37\}$ for a total of $2!\cdot 2!\cdot 4! = 96$ total choices of $\sigma$.

11.
In the diagram shown at right, how many ways are there to remove at least one edge so that some circle with an "A" and some circle with a "B" remain connected?




**Answer:** $4095$

**Solution:** Since the dual problem is identical, exactly half the arrangements work (there's a path either from top to bottom or from left to right, but not both). So our answer is $\frac{1}{2}(2^{13})-1 = 4095$.

12. Let $\mathcal{S}$ be the set of the $125$ points in three-dimension space of the form $(x,y,z)$ where $x$, $y$, and $z$ are integers between $1$ and $5$, inclusive. A family of snakes lives at the point $(1,1,1)$, and one day they decide to move to the point $(5,5,5)$.

Snakes may slither only in increments of $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Given that at least one snake has slithered through each point of $\mathcal{S}$ by the time the entire family has reached $(5, 5, 5)$, what is the smallest number of snakes that could be in the family?

**Answer:** $19$

**Solution:** Note that every possible path goes through 13 cubes: one at each distance (0,1,2,3,...,12) from the bottom, leftmost, backmost cube (using a taxicab metric). Also note that there are 19 cubes at distance 6 (they form a hexagon with rows containing 3,4,5,4,3 cubes respectively from top to bottom). Therefore it takes at least 19 snakes to complete this task. It is trivial to come up with such an example.