.... Mathathon Round 1 (2 points each) ....

1. A circle is inscribed inside a square such that the cube of the radius of the circle is numerically equal to the perimeter of the square. What is the area of the circle?

Answer: 8π

Solution: Since the circle is inscribed inside the square, the diameter $2r$ of the circle is equal to the side length of the square. Thus we are given that $r^3 = 4(2r)$, so $r^2 = 8$ and the area of the circle is $\pi r^2 = 8\pi$.

2. If the coefficient of $z^k y^k$ is 252 in the expression $(z + y)^{2k}$, find k.

Answer: 5

Solution: By the Binomial Theorem, we know that the coefficient of $z^k y^k$ is $\binom{2k}{k}$. Therefore, we solve

$$\binom{2k}{k} = \frac{(2k)!}{k!k!} = 252.$$

Some simple checking gives $k = 5$.

3. Let $f(x) = \frac{4x^4-2x^3-x^2-3x-2}{x^4-x^3+x^2-x-1}$ be a function defined on the real numbers where the denominator is not zero. The graph of f has a horizontal asymptote. Compute the sum of the x-coordinates of the points where the graph of f intersects this horizontal asymptote. If the graph of f does not intersect the asymptote, write 0.

Answer: $\frac{5}{2}$

Solution: The asymptote is $y = 4$. Compute $f(x)-4$, and then find the roots of the numerator. We have $\frac{(x-2)(x-1)(2x+1)}{(x-1)x(x^2+1)-1} = 0$, so $x = 1, 2, -\frac{1}{2}$, and the sum is $\frac{5}{2}$.

.... Mathathon Round 2 (3 points each) ....

4. How many 5-digit numbers have strictly increasing digits? For example, 23789 has strictly increasing digits, but 23889 and 23869 do not.

Answer: 126

Solution: For every choice of 5 distinct digits, we can construct exactly one such number. However, we cannot use the digit 0 because, for example, 02357 is not a 5-digit number. We must choose 5 digits from 9 digits. The number of possible combinations is $\binom{9}{5} = \frac{9!}{5!4!} = 126$.

5. Let

$$y = \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{5 + \cfrac{1}{9 + \cfrac{1}{5 + \cdots}}}}}$$

If y can be represented as $\dfrac{a\sqrt{b}+c}{d}$, where b is not divisible by any squares, and the greatest common divisor of a and d is 1, find the sum $a + b + c + d$.

Answer: 135

Solution: First, we use the classic method of solving just $\cfrac{1}{9+\cfrac{1}{5+\cfrac{1}{9+\cfrac{1}{5+\cdots}}}}$, by setting up the equation

$$\cfrac{1}{9+\cfrac{1}{5+x}} = x.$$

Solving this gives us the fraction

$$\frac{x+5}{9x+46} = x \Rightarrow 9x^2 + 46x = x + 5.$$

Solving this gives us that

$$x = \frac{-45 \pm \sqrt{45^2 + 4 \cdot 9 \cdot 5}}{2 \cdot 9} = -\frac{5}{2} \pm \frac{7\sqrt{5}}{6}.$$

Since x represents a nested fraction, it must be positive in value, so we know that

$$x = -\frac{5}{2} + \frac{7\sqrt{5}}{6}.$$

Finally, plugging back into the original question, we see that

$$y = \frac{1}{1 + (-\frac{5}{2} + \frac{7\sqrt{5}}{6})} = \frac{21\sqrt{5}+27}{82},$$

so the answer is $21 + 5 + 27 + 82 = 136$.

6. "Counting" is defined as listing positive integers, each one greater than the previous, up to (and including) an integer n. In terms of n, write the number of ways to count to n.

Answer: 2^{n-1}

Solution: Note that we need to have n in our count. We have $n-1$ integers before that, and for each one we have two choices – write it down or skip it. So there are 2^{n-1} total ways.

.... Mathathon Round 3 (4 points each) ....

7. Suppose p, q, $2p^2 + q^2$, and $p^2 + q^2$ are all prime numbers. Find the sum of all possible values of p.

Answer: 2

Solution: Note that $(2p^2 + q^2) - (p^2 + q^2) = p^2$. $p^2 + q^2 > 1$, so the difference between two odd numbers is even. Therefore p is an even prime, so $p = 2$.

8. Let $r(d)$ be a function that reverses the digits of the 2-digit integer d. What is the smallest 2-digit positive integer N such that for some 2-digit positive integer n and 2-digit positive integer $r(n)$, N is divisible by n and $r(n)$, but not by 11?

Answer: 84

Solution: We want to minimize $\text{lcm}(r(n), n)$, which is at least twice the smaller of the two. $n = 24$ or 42 both work, and give 84. If either digit ≥ 4 then the number is > 84, so we are done.

9. What is the period of the function $y = (\sin(3\theta) + 6)^2 - 10(\sin(3\theta) + 7) + 13$?

Answer: $\frac{2\pi}{3}$

Solution: $(\sin(3\theta)+6)^2 - 10(\sin(3\theta)+7)+13 = \sin^2(3\theta)+12\sin(3\theta)+36-10\sin(3\theta)-70+13$, $y = \sin^2(3\theta)+2\sin(3\theta)+1-22$, $y = (\sin(3\theta)+1)^2 - 22$. The function $\sin(3\theta)$ repeats every $\frac{2\pi}{3}$

. . . . Mathathon Round 4 (6 points each) . . . .

10. Three numbers a, b, c are given by $a = 2^2(\sum_{i=0}^{2} 2^i)$, $b = 2^4(\sum_{i=0}^{4} 2^i)$, and $c = 2^6(\sum_{i=0}^{6} 2^i)$. u, v, w are the sum of the divisors of a, b, c respectively, yet excluding the original number itself. What is the value of $a + b + c - u - v - w$

Answer: 0

Solution: It should be noted that a, b, c are of the form $p(2^n)$ where p is a prime and n is an integer. The sum of the divisors is $\sum_{i=0}^{n} 2^i + p(\sum_{i=0}^{n-1} 2^i)$ when excluding the original number itself. Since $p = \sum_{i=0}^{n} 2^i$, the value becomes $p(2^n)$. Thus the sum of the divisors of the original number excluding the original number itself equals the original number and $a+b+c-u-v-w = 0$.

11. Compute $\sqrt{6 - \sqrt{11}} - \sqrt{6 + \sqrt{11}}$.

Answer: $-\sqrt{2}$

Solution: Square the expression and simplify to get 2. The expression is negative since the first term has $6 - \sqrt{11}$ and the second term has $6 + \sqrt{11}$, so we take the negative square root to get the answer $-\sqrt{2}$.

12. Let $a_0, a_1, \ldots a_n$ be such that $a_n \neq 0$ and

$$(1 + x + x^3)^{341}(1 + 2x + x^2 + 2x^3 + 2x^4 + x^6)^{342} = \sum_{i=0}^{n} a_i x^i.$$

Find the number of odd numbers in the sequence $a_0, a_1, \ldots a_n$.

Answer: 3

Solution: The expression reduces to $(1 + x + x^3)^{2^{10}}$. The number of odd coefficients doesn't change each time you square this, so there's still 3 after squaring 10 times.

. . . . Mathathon Round 5 (7 points each) . . . .

13. How many ways can we form a group with an odd number of members (plural) from 99 people? Express your answer in the form $a^b + c$, where a, b, and c are integers and a is prime.

Answer: $2^{98} - 99$

Solution: We want to find the sum

$$\binom{99}{1} + \binom{99}{3} + \ldots + \binom{99}{99}$$

Since we can put the number of subsets with an odd number of elements in 1-1 correspondence with the number of subsets with an even number of elements, for any set, this sum is equal to

$$\frac{1}{2}\left(\binom{99}{0} + \binom{99}{1} + \ldots + \binom{99}{99}\right) = \frac{1}{2}(2^{99}) = 2^{98}$$

. If we require that every group has more than one member, then our answer is $2^{98} - \binom{99}{1} = 2^{98} - 99$.

14. A cube is inscibed in a right circular cone such that the ratio of the height of the cone to the radius is 2:1. Compute the fraction of the cone's volume that the cube occupies.

Answer: $\frac{12}{\pi}(\sqrt{2} - 1)^3$

Solution: Let h be the height of the pyramid, r be the radius, and s be the side length of the square. We see that $2r = h$. We can solve for the desired ratio by considering similar triangles in the cross sectional plane. Then $\frac{2(h-s)}{s\sqrt{2}} = \frac{h}{r} = 2$.

15. Let $F_0 = 1, F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$. Let $P(x) = \sum_{k=0}^{99} x^{F_k}$. The remainder when $P(x)$ is divided by $x^3 - 1$ can be expressed as $ax^2 + bx + c$. Find $2a + b$.

Answer: 112

Solution: Counting the Fibonacci sequence mod 3 gives a period of 8: 1, 1, 2, 0, 2, 2, 1, 0 and repeats. We know a counts the number of terms with powers 2 (mod 3) in the first 100 terms, b counts the number of terms 1 (mod 3), and c counts the number of terms 0 (mod 3). We can compute using that periodic sequence that $(a, b, c) = (37, 38, 25)$.

.... Mathathon Round 6 (8 points each) ....

16. If $|x| < \frac{1}{4}$ and

$$X = \sum_{N=0}^{\infty} \sum_{n=0}^{N} \binom{N}{n} x^{2n} (2x)^{N-n},$$

then write X in terms of x without any summation or product symbols (and without an infinite number of '+'s, etc.).

Answer: $\dfrac{1}{1 - 2x - x^2}$

Solution: The inner sum is the binomial expansion $(2x + x^2)^n$. This is a geometric series with first term 1 and common ratio $2x + x^2$. The result is as stated in the answer (noting that the bound on x is more than sufficient).

17. Ankit finds a quite peculiar deck of cards in that each card has n distinct symbols on it and any two cards chosen from the deck will have exactly one symbol in common. The cards are guaranteed to not have a certain symbol which is held in common with all the cards. Ankit decides to create a function $f(n)$ which describes the maximum possible number of cards in a set given the previous constraints. What is the value of $f(10)$?

Answer: 91

Solution: One can start by looking at a particular card with 10 unique symbols on it. A "'branch"' of nine cards can be created all having in common a particular symbol with the starting card. The branch cannot be any longer due to the fact that other cards on other branches would not be able to have a symbol in common with all the cards in this branch. Thus there can exist ten branches with nine cards in each branch along with the one starting card.The function $f(n) = n(n-1)+1$. Thus $f(10) = 10(9) + 1 = 91$.

18. Dietrich is playing a game where he is given three numbers a, b, c which range from $[0, 3]$ in a continuous uniform distribution. Dietrich wins the game if the maximum distance between any two numbers is no more than 1. What is the probability Dietrich wins the game?

Answer: $\frac{7}{27}$

Solution: A three dimensional graph with coordinate axes x,y, and z could be used to represent combination of values of a, b, and c. The probability will be symmetric about plane y=x so one can thus consider the probability when $y \geq x$. When $y \geq x$, $(y - 1) \leq z \leq (x + 1)$. For $1 \leq x \leq 2$, $(y - 1) \leq z \leq (x + 1)$ which produces a trapezoidal prism. For $2 \leq x \leq 3$, $(y - 1) \leq z \leq 3$ which produces a triangular prism along with a pyramid. For $0 \leq x \leq 1$, $0 \leq z \leq (x + 1)$ which produces a rectangular prism along with a pyramid. The sum of the volumes is: $\frac{3}{2} + \frac{1}{2} + \frac{1}{6} + 1 + \frac{1}{3} = \frac{7}{2}$. The probability is $\frac{\frac{7}{2}}{\frac{27}{2}} = \frac{7}{27}$

.... Mathathon Round 7 (9 points each) ....

19. Consider f defined by

$$f(x) = x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6.$$

How many tuples of positive integers $(a_1, a_2, a_3, a_4, a_5, a_6)$ exist such that $f(-1) = 12$ and $f(1) = 30$?

Answer: 4788

Solution: $1 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 = 12 = f(-1)$ and $1 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 30 = f(1)$. Assume a_i's positive, so $2 + 2a_2 + 2a_4 + 2a_6 = 42$, so $a_2 + a_4 + a_6 = 20$ and $a_1 + a_3 + a_5 = 9$. Then we have a balls and bins problem, where there are $\binom{19}{2}\binom{8}{2} = 4788$ total ways.

20. Let a_n be the number of permutations of the numbers $S = \{1, 2, \ldots, n\}$ such that for all k with $1 \le k \le n$, the sum of k and the number in the kth position of the permutation is a power of 2. Compute $a_{2^0} + a_{2^1} + a_{2^2} \cdots + a_{2^{20}}$.

Answer: 21

Solution: We prove that $a_n = 1$ for all n by strong induction. Clearly $a_1 = 1$. Suppose that $a_n = 1$ for all values less than n. Let $n = 2^m + r$ for $0 \le r < 2^m$. Consider $S = \{2^m, 2^m + 1, \ldots, 2^m + r\}$. For each s in S, $\pi(s) + s$ has to be 2^{m+1}, since every other power of 2 is either too high or too low. Therefore, there is a unique spot determined for everything, by employing the inductive hypothesis. Finally, we have exponents from 0 to 20, or 21 numbers.

21. A 4-dimensional hypercube of edge length 1 is constructed in 4-space with its edges parallel to the coordinate axes and one vertex at the origin. Its coordinates are given by all possible permutations of $(0, 0, 0, 0), (1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0)$, and $(1, 1, 1, 1)$. The 3-dimensional hyperplane given by $x+y+z+w = 2$ intersects the hypercube at 6 of its vertices. Compute the 3-dimensional volume of the solid formed by the intersection.

Answer: $\frac{4}{3}$

Solution: The hyperplane intersects the hypercube at the vertices given by all the permutations of $(1, 1, 0, 0)$. We will show that the 3-dimensional solid formed by the intersection is a regular octahedron. First, note that the center of the solid is the same as the center of the hypercube, namely $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The distance from the center to each of the 6 vertices is the same. We can show that the four vertices (1,1,0,0), (1,0,1,0), (0,0,1,1), and (0,1,0,1) form a square, as well as the vertices (1,1,0,0), (1,0,0,1), (0,0,1,1), and (0,1,1,0) as well as the vertices (1,0,0,1), (1,0,1,0), (0,1,1,0), and (0,1,0,1). Note that each set of four vertices shares two elements with each other set. Thus the six points form the vertices of a regular octahedron, so the cross section described in the problem is a regular octahedron. The edge length of this octahedron is $\sqrt{1^2 + 1^2} = \sqrt{2}$. There are at least two ways to find the volume from this point on: 1. Use the volume formula for a regular octahedron: $\frac{\sqrt{2}}{3}s^3$, where s is the side length. Then the volume of the octahedron is $\frac{\sqrt{2}}{3}(\sqrt{3})^3 = \frac{4}{3}$. 2. A regular octahedron can be divided into eight pyramids by planes through the four squares formed by the vertices. Each of these pyramids has a volume of $\frac{1}{3}(1)(\frac{1}{2})(1)(1) = \frac{1}{6}$. Then the octahedron's volume is $8(\frac{1}{6}) = \frac{4}{3}$.