## Mathathon Round 1 (2 points each)

1. A circle is inscribed inside a square such that the cube of the radius of the circle is numerically equal to the perimeter of the square. What is the area of the circle?

2. If the coefficient of $z^k y^k$ is 252 in the expression $(z+y)^{2k}$, find $k$.

3. Let $f(x) = \frac{4x^4-2x^3-x^2-3x-2}{x^4-x^3+x^2-x-1}$ be a function defined on the real numbers where the denominator is not zero. The graph of $f$ has a horizontal asymptote. Compute the sum of the $x$-coordinates of the points where the graph of $f$ intersects this horizontal asymptote. If the graph of $f$ does not intersect the asymptote, write 0.

## Mathathon Round 2 (3 points each)

4. How many 5-digit numbers have strictly increasing digits? For example, 23789 has strictly increasing digits, but 23889 and 23869 do not.

5. Let

$$y = \cfrac{1}{1+\cfrac{1}{9+\cfrac{1}{5+\cfrac{1}{9+\cfrac{1}{5+\cdots}}}}}$$

If $y$ can be represented as $\dfrac{a\sqrt{b}+c}{d}$, where $b$ is not divisible by any squares, and the greatest common divisor of $a$ and $d$ is 1, find the sum $a+b+c+d$.

6. "Counting" is defined as listing positive integers, each one greater than the previous, up to (and including) an integer $n$. In terms of $n$, write the number of ways to count to $n$.

## Mathathon Round 3 (4 points each)

7. Suppose $p$, $q$, $2p^2+q^2$, and $p^2+q^2$ are all prime numbers. Find the sum of all possible values of $p$.

8. Let $r(d)$ be a function that reverses the digits of the 2-digit integer $d$. What is the smallest 2-digit positive integer $N$ such that for some 2-digit positive integer $n$ and 2-digit positive integer $r(n)$, $N$ is divisible by $n$ and $r(n)$, but not by 11?

9. What is the period of the function $y = (\sin(3\theta)+6)^2 - 10(\sin(3\theta)+7)+13$?

## Mathathon Round 4 (6 points each)

10. Three numbers $a, b, c$ are given by $a = 2^2(\sum_{i=0}^{2} 2^i)$, $b = 2^4(\sum_{i=0}^{4} 2^i)$, and $c = 2^6(\sum_{i=0}^{6} 2^i)$. $u, v, w$ are the sum of the divisors of $a, b, c$ respectively, yet excluding the original number itself. What is the value of $a+b+c-u-v-w$

11. Compute $\sqrt{6-\sqrt{11}} - \sqrt{6+\sqrt{11}}$.

12. Let $a_0, a_1, \ldots a_n$ be such that $a_n \neq 0$ and

$$(1+x+x^3)^{341}(1+2x+x^2+2x^3+2x^4+x^6)^{342} = \sum_{i=0}^{n} a_i x^i.$$

Find the number of odd numbers in the sequence $a_0, a_1, \ldots a_n$.

## Mathathon Round 5 (7 points each)

13. How many ways can we form a group with an odd number of members (plural) from 99 people? Express your answer in the form $a^b + c$, where $a, b$, and $c$ are integers and $a$ is prime.

14. A cube is inscibed in a right circular cone such that the ratio of the height of the cone to the radius is 2:1. Compute the fraction of the cone's volume that the cube occupies.

15. Let $F_0 = 1, F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$. Let $P(x) = \sum_{k=0}^{99} x^{F_k}$. The remainder when $P(x)$ is divided by $x^3 - 1$ can be expressed as $ax^2 + bx + c$. Find $2a + b$.

## Mathathon Round 6 (8 points each)

16. Ankit finds a quite peculiar deck of cards in that each card has $n$ distinct symbols on it and any two cards chosen from the deck will have exactly one symbol in common. The cards are guaranteed to not have a certain symbol which is held in common with all the cards. Ankit decides to create a function $f(n)$ which describes the maximum possible number of cards in a set given the previous constraints. What is the value of $f(10)$?

17. If $|x| < \frac{1}{4}$ and

$$X = \sum_{N=0}^{\infty} \sum_{n=0}^{N} \binom{N}{n} x^{2n} (2x)^{N-n},$$

then write $X$ in terms of $x$ without any summation or product symbols (and without an infinite number of '+'s, etc.).

18. Dietrich is playing a game where he is given three numbers a, b, c which range from $[0, 3]$ in a continuous uniform distribution. Dietrich wins the game if the maximum distance between any two numbers is no more than 1. What is the probability Dietrich wins the game?

## Mathathon Round 7 (9 points each)

19. Consider $f$ defined by

$$f(x) = x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6.$$

How many tuples of positive integers $(a_1, a_2, a_3, a_4, a_5, a_6)$ exist such that $f(-1) = 12$ and $f(1) = 30$?

20. Let $a_n$ be the number of permutations of the numbers $S = \{1, 2, \ldots, n\}$ such that for all $k$ with $1 \leq k \leq n$, the sum of $k$ and the number in the $k$th position of the permutation is a power of 2. Compute $a_1 + a_2 + a_4 + a_8 + \cdots + a_{1048576}$.

21. A 4-dimensional hypercube of edge length 1 is constructed in 4-space with its edges parallel to the coordinate axes and one vertex at the origin. Its coordinates are given by all possible permutations of $(0, 0, 0, 0), (1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0)$, and $(1, 1, 1, 1)$. The 3-dimensional hyperplane given by $x + y + z + w = 2$ intersects the hypercube at 6 of its vertices. Compute the 3-dimensional volume of the solid formed by the intersection.