Math Majors of America Tournament for High Schools
2018 Individual Test Solutions

1. Five friends arrive at a hotel which has three rooms. Rooms A and B hold two people each, and room C holds one person. How many different ways could the five friends lodge for the night?

Answer: 30

Solution: There are 5 choices for who gets room C . After that, there are $\binom{4}{2} = 6$ ways to assign 2 people to room A. In total, there are $(5)(6) = 30$ possibilities.

2. The set of numbers $\{1, 3, 8, 12, x\}$ has the same average and median. What is the sum of all possible values of x? (Note that x is not necessarily greater than 12.)

Answer: 13

Solution: The median could be 3, 8, or x. In the first case, we get $24 + x = 5 \cdot 3 \to x = -9$. In the second case, we get $24 + x = 5 \cdot 8 \to x = 16$. In the third case, we get $24 + x = 5 \cdot x \to x = 6$. The sum of these solutions is $-9 + 16 + 6 = 13$.

3. How many four-digit numbers $\overline{ABCD}$ are there such that the three-digit number $\overline{BCD}$ satisfies $\overline{BCD} = \frac{1}{6}\overline{ABCD}$? (Note that A must be nonzero.)

Answer: 4

Solution: Write $\overline{ABCD} = 1000A + \overline{BCD} = 6\overline{BCD}$, so $1000A = 5\overline{BCD}$ and $\overline{BCD}$ must be 200, 400, 600, or 800. The four corresponding possibilities for $\overline{ABCD}$ are 1200, 2400, 3600, and 4800.

4. Find the smallest positive integer n such that n leaves a remainder of 5 when divided by 14, n^2 leaves a remainder of 1 when divided by 12, and n^3 leaves a remainder of 7 when divided by 10.

Answer: 103

Solution: $n^2 \equiv 1(12) \to n \equiv 1, 5, 7, 11(12)$, and $n^3 \equiv 7(10) \to n \equiv 3(10)$. By trying small values, 33 is the smallest positive integer satisfying the conditions on n and n^3. Then add multiples of 70 until the n^2 condition is satisfied. $33 + 70 = 103 \equiv 7(12)$ works.

5. In rectangle $ABCD$, let E lie on $\overline{CD}$, and let F be the intersection of $\overline{AC}$ and $\overline{BE}$. If the area of $\triangle ABF$ is 45 and the area of $\triangle CEF$ is 20, find the area of the quadrilateral $ADEF$.

Answer: 55

Solution: Let G lie on $\overline{BC}$ such that $\overline{BC}$ is perpendicular to $\overline{FG}$. Since $\triangle ABF \sim \triangle CEF$, we see that $\overline{BG} = \sqrt{45}h$ and $\overline{CG} = \sqrt{20}h$ for some h. Then we know that $\frac{\overline{AB} \cdot \sqrt{45}h}{2} = 45 \to \overline{AB} = \frac{2\sqrt{45}}{h}$. Thus the area of rectangle $ABCD$ is $\frac{2\sqrt{45}}{h} \cdot (\sqrt{45} + \sqrt{20})h = 90 + 60 = 150$. So the area of quadrilateral $ADEF$ is $150 - 20 - \frac{2\sqrt{45} \cdot (\sqrt{20} + \sqrt{45})}{2} = 130 - (30 + 45) = 55$.

6. If x and y are integers and $14x^2y^3 - 38x^2 + 21y^3 = 2018$, what is the value of x^2y?

Answer: 50

Solution: We can rewrite this as $(2x^2 + 3)(7y^3 - 19) + 57 = 2018$. This means $(2x^2 + 3)(7y^3 - 19) = 1961 = 37 * 53$. So $2x^2 + 3 \in \{1, 37, 53, 1961\}$. Of these options, only 53 is 3 more than twice a square, so $x = \pm 5$. Then $7y^3 - 19 = 37$ and $y = 2$.

7. A, B, C, D all lie on a circle with $\overline{AB} = \overline{BC} = \overline{CD}$. If the distance between any two of these points is a positive integer, what is the smallest possible perimeter of quadrilateral $ABCD$?

Answer: 17

Solution: Let $\overline{AB} = \overline{BC} = \overline{CD} = x$ and $\overline{AD} = y$. Note that $ABCD$ is an isosceles trapezoid so that $\overline{AC} = \overline{BD} = z$. By Ptolemy's theorem, we have $(\overline{AB})(\overline{CD}) + (\overline{BC})(\overline{AD}) = (\overline{AC})(\overline{BD})$. So the integers x, y, and z satisfy $(x)(x) + (x)(y) = (z)(z)$ i.e. $x(x+y) = z^2$. Also, by the triangle inequality, $y < 3x$. Therefore, z^2 factors as the product of two distinct positive integers such that smaller integer is strictly more than $\frac{1}{4}$ of the larger integer. 1^2 cannot be factored as the product of distinct integers. 2^2 can only be factored as $1 * 4$. 3^2 can only be factored as $1 * 9$. 4^2 can only be factored as $1 * 16$ or $2 * 8$. 5^2 can only be factored as $1 * 25$. 6^2 can be factored as $4 * 9$, which gives $x = 4$, $y = 5$ as a solution. Therefore a perimeter of $3x + y = 17$ is possible. If $x \leq 3$, then $y < 9$ and $z^2 < 3(3+9) = 36$, which is not possible, so $x \geq 4$. If $x \geq 5$ and the perimeter is less than 17, then $x = 5$, $y = 1$ is the only possibility, and this leads to $z^2 = 30$, which is not a perfect square. Therefore $x = 4$ yields the minimum possible perimeter. If $y < 5$, then $z^2 < 36$ is not possible, so $x = 4$, $y = 5$ does in fact yield the minimum possible perimeter of 17.

8. Compute

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\cos^2(n) + n\sin^2(m)}{3^{m+n}(m+n)}.$$

Answer: $\frac{1}{8}$

Solution: Let $x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\cos^2(n)+n\sin^2(m)}{3^{m+n}(m+n)}$. Switching m and n leaves the sum unchanged, so $x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n\cos^2(m)+m\sin^2(n)}{3^{m+n}(m+n)}$ and $2x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m(\cos^2(n)+\sin^2(n))+n(\sin^2(m)+\cos^2(m))}{3^{m+n}(m+n)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m+n}{3^{m+n}(m+n)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}3^{-m-n} = (\sum_{m=1}^{\infty}3^{-m})^2 = (\frac{\frac{1}{3}}{1-\frac{1}{3}})^2 = (\frac{1}{2})^2 = \frac{1}{4} \rightarrow x = \frac{1}{8}$.

9. Diane has a collection of weighted coins with different probabilities of landing on heads, and she flips nine coins sequentially according to a particular set of rules. She uses a coin that always lands on heads for her first and second flips, and she uses a coin that always lands on tails for her third flip. For each subsequent flip, she chooses a coin to flip as follows: if she has so far flipped a heads out of b total flips, then she uses a coin with an $\frac{a}{b}$ probability of landing on heads. What is the probability that after all nine flips, she has gotten six heads and three tails?

Answer: $\frac{5}{28}$

Solution: There are $\binom{6}{2} = 15$ orders for the remaining 4 heads and 2 tails. For each of these cases, when we find the product of the probabilties of all 6 of the flips landing as needed, we end up with 2, 3, 4, 5, 1, 2 in the numerator (in some order) and 3, 4, 5, 6, 7, and 8 in the denominator. This gives $\frac{(2)(3)(4)(5)(2)}{(3)(4)(5)(6)(7)(8)} = \frac{1}{84}$. So the probability of one of the 15 cases occurring is $\frac{15}{84} = \frac{5}{28}$.

10. For any prime number p, let S_p be the sum of all the positive divisors of $37^p p^{37}$ (including 1 and $37^p p^{37}$). Find the sum of all primes p such that S_p is divisible by p.

Answer: 19

Solution: Let $\sigma(n)$ be the sum of the positive divisors of n. For a prime p, we see that $\sigma(p^r) = 1 + p + p^2 + \cdots + p^r = \frac{p^{r+1}-1}{p-1}$, and if m and n are relatively prime, then $\sigma(mn) = \sigma(m)\sigma(n)$. Hence for $p \neq 37$, $S_p = \sigma(37^p)\sigma(p^{37}) = \frac{(37^{p+1}-1)(p^{38}-1)}{(36)(p-1)}$. The second term in the numerator is never divisible by p. By Fermat's Little Theorem, the first term is $37^2 - 1 = 1368 = 2^3 \cdot 3^2 \cdot 19$ modulo p, which is divisible by p only for $p = 2, 3, 19$. Check these cases. If $p = 2$, we want to know if $37^3 - 1$ is divisible by 8 (to cancel all the 2's in the denominator). Modulo 8: $37^3 - 1 \sim 5^3 - 1 = 124 \sim 4 \neq 0$, so $p = 2$ doesn't work. If $p = 3$, take everything modulo 27 for the same reason. We get $37^4 - 1 \sim 10^4 - 1 = 9999 = 9(1111)$ is not a multiple of 27. If $p = 19$, we don't have to check anything else because there's no 19 in the denominator. Finally, compute $S_{37} = \sigma(37^2) = \frac{37^3-1}{36}$ is not divisible by 37. So the answer is just 19.

11. Six people are playing poker. At the beginning of the game, they have 1, 2, 3, 4, 5, and 6 dollars, respectively. At the end of the game, nobody has lost more than a dollar, and each player has a distinct nonnegative integer dollar amount. (The total amount of money in the game remains constant.) How many distinct finishing rankings (i.e. lists of first place through sixth place) are possible?

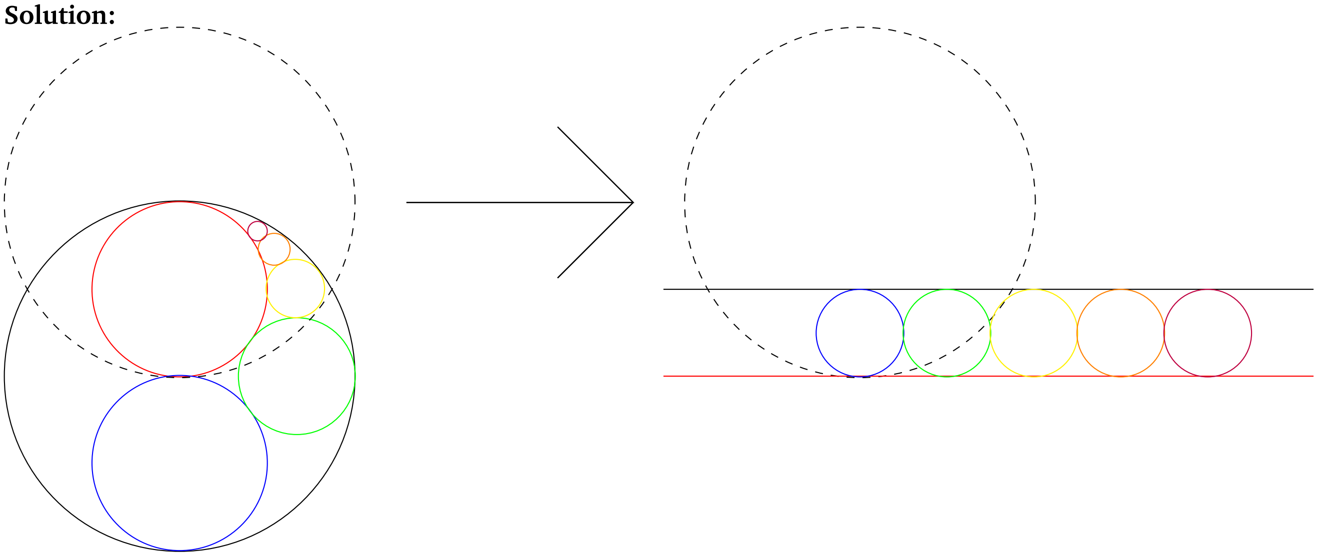
Answer: 34

Solution: We consider two cases. In the first case, nobody loses more than one place in the rankings. To determine the players who lose a ranking, we can choose any subset of the five players who started with at least \$2, then there is exactly one way to rank the remaining players. There are $2^5 = 32$ such subsets. In the second case, at least one player loses two places in the rankings. Since nobody loses more than \$1, essentially, we may imagine our orignal 6 players each losing \$1 and then redistributing the extra \$6 in a way that causes a player to be 2 ranks lower than they were before. Since the players cannot lose anymore money, this means distributing the \$6 such that two players who had less than some player to start, finish with more than that player. Note that for one player to overcome another, they must gain at least \$2 if they started with \$1 less, and \$3 if they started with \$2 less (and even more otherwise). Furthermore for them to end with different totals, one must gain an additional \$1. Note that this provides just enough for the players beginning ranked second and third to overcome the player originally ranked first, but not enough for any other combination of players to overcome the first player. It is also not enough for two players to overcome any players other than the first ranked player since we would need more money in order to prevent ties. Therefore we are left with only two possibilities in this case (when the second ranked player ends up first and the third ranked player ends in second, and the possibility where the third ranked player ends in first and the second ranked player ends in second. Therefore there are $32+2=34$ total distinct finishing rankings.

12. Let C_1 be a circle of radius 1, and let C_2 be a circle of radius $\frac{1}{2}$ internally tangent to C_1. Let $\{\omega_0, \omega_1, \dots\}$ be an infinite sequence of circles, such that ω_0 has radius $\frac{1}{2}$ and each ω_k is internally tangent to C_1 and externally tangent to both C_2 and ω_{k+1}. (The ω_k's are mutually distinct.) What is the radius of ω_{100}?

Answer: $\frac{1}{10002}$

Solution:



Consider an inversion about a circle with radius 1 centered at the point of tangency between C_1 and C_2. (An inversion is performed by mapping the point at distance d from the center of the circle to the point at distance $\frac{r^2}{d}$ in the same direction. Inversions preserve circles unless they pass through the center of inversion in which case they are transformed into lines.) We end up with a diagram consisting of a line of circles $\omega_0, \omega_1, \dots,$ all with radius $\frac{1}{4}$. ω_0 has its center $\frac{3}{4}$ away from the center of inversion, and the other circles continue in a perpendicular direction. Therefore, the inversion of ω_n has its center a distance $\sqrt{(\frac{3}{4})^2 + (\frac{n}{2})^2}$ from from the center of inversion. Therefore the distance from the point of inversion to the closest point in the inverted ω_n is $\sqrt{(\frac{3}{4})^2 + (\frac{n}{2})^2} - \frac{1}{4}$ and the distance to the furthest point is $\sqrt{(\frac{3}{4})^2 + (\frac{n}{2})^2} + \frac{1}{4}$. Therefore in the original ω_n, these distances are $\frac{1}{\sqrt{(\frac{3}{4})^2+(\frac{n}{2})^2}-\frac{1}{4}}$ and $\frac{1}{\sqrt{(\frac{3}{4})^2+(\frac{n}{2})^2}+\frac{1}{4}}$, respectively. So the diameter of ω_n is $\frac{1}{\sqrt{(\frac{3}{4})^2+(\frac{n}{2})^2}-\frac{1}{4}} - \frac{1}{\sqrt{(\frac{3}{4})^2+(\frac{n}{2})^2}+\frac{1}{4}} = (4)\left(\frac{1}{\sqrt{9+4n^2}-1} - \frac{1}{\sqrt{9+4n^2}+1}\right) = (4)\left(\frac{(\sqrt{9+4n^2}+1)-(\sqrt{9+4n^2}-1)}{(9+4n^2)-1}\right) = \frac{2}{n^2+2}$. And the radius of ω_n is $\frac{1}{n^2+2}$. Plugging in $n=100$ gives $\frac{1}{10002}$.