Tiebreaker Round

Girls in Math at Yale

February 27, 2021

Problem 1 *In their class Introduction to Ladders at Greendale Community College, Jan takes four tests. They realize that their test scores in chronological order form a strictly increasing arithmetic progression with integer terms, and that the average of those scores is an integer greater than or equal to 94. How many possible combinations of test scores could they have had? (Test scores at Greendale range between 0 and 100, inclusive.)*

Problem 2 *Suppose that A and B are digits between 1 and 9 such that*

$$0.\overline{ABABAB\cdots} + B \cdot (0.\overline{AAA\cdots}) = A \cdot (0.\overline{B1B1B1\cdots}) + 1.$$

Find the sum of all possible values of $10A + B$.

Problem 3 *Let ABC be an isosceles right triangle with $m\angle ABC = 90°$. Let D and E lie on segments $\overline{AC}$ and $\overline{BC}$, respectively, such that triangles $\triangle ADB$ and $\triangle CDE$ are similar and $DE = EB$. If $\frac{AC}{AD} = 1 + \frac{\sqrt{a}}{b}$ with a, b positive integers and a squarefree, then find $a + b$.*

Problem 4 *Five bowling pins $P_1, P_2, \cdots, P_5$ are lined up in a row. Each turn, Jemma picks a pin at random from the standing pins, and throws a bowling ball at that pin; that pin and each pin directly adjacent to it are knocked down. If the expected value of the number of turns Jemma will take to knock down all the pins is $\frac{a}{b}$ where a and b are relatively prime, find $a + b$. (Pins P_i and P_j are adjacent if and only if $|i - j| = 1$.)*

Problem 5 *How many terms in the expansion of*

$$(1 + x + x^2 + x^3 + \cdots + x^{2021})(1 + x^2 + x^4 + x^6 + \cdots + x^{4042})$$

have coefficients equal to 1011?

Problem 6 *Suppose $f(x)$ is a monic quadratic polynomial with distinct nonzero roots p and q, and suppose $g(x)$ is a monic quadratic polynomial with roots $p + \frac{1}{q}$ and $q + \frac{1}{p}$. If we are given that $g(-1) = 1$ and $f(0) \neq -1$, then there exists some real number r that must be a root of $f(x)$. Find r.*