Mixer Round

Girls in Math at Yale

February 27, 2021

Problem 1 *Find the number of ordered triples* (a, b, c) *satisfying*

- a, b, c *are are single-digit positive integers, and*
- $a \cdot b + c = a + b \cdot c$.

Problem 2 *In their class Introduction to Ladders at Greendale Community College, Jan takes four tests. They realize that their test scores in chronological order form an increasing arithmetic progression with integer terms, and that the average of those scores is an integer greater than or equal to* 94*. How many possible combinations of test scores could they have had? (Test scores at Greendale range between* 0 *and* 100*, inclusive.)*

Problem 3 *Suppose that* $a + \frac{1}{b} = 2$ *and* $b + \frac{1}{a} = 3$*. If* $\frac{a}{b} + \frac{b}{a}$ *can be expressed as* $\frac{p}{q}$ *in simplest terms, find* $p + q$.

Problem 4 *Suppose that* A *and* B *are digits between* 1 *and* 9 *such that*

$$0.\overline{ABABAB\cdots} + B \cdot (0.\overline{AAA\cdots}) = A \cdot (0.\overline{B1B1B1\cdots}) + 1.$$

Find the sum of all possible values of $10A + B$.

Problem 5 *Let* ABC *be an isosceles right triangle with* $m\angle ABC = 90^\circ$*. Let* D *and* E *lie on segments* $\overline{AC}$ *and* $\overline{BC}$*, respectively, such that triangles* $\triangle ADB$ *and* $\triangle CDE$ *are similar and* $DE = EB$*. If* $\frac{AC}{AD} = 1 + \frac{\sqrt{a}}{b}$ *with* a, b *positive integers and* a *squarefree, then find* $a + b$.

Problem 6 *Five bowling pins* $P_1, P_2, \cdots, P_5$ *are lined up in a row. Each turn, Jemma picks a pin at random from the standing pins, and throws a bowling ball at that pin; that pin and each pin directly adjacent to it are knocked down. If the expected value of the number of turns Jemma will take to knock down all the pins is* $\frac{a}{b}$ *where* a *and* b *are relatively prime, find* $a + b$*. (Pins* P_i *and* P_j *are adjacent if and only if* $|i - j| = 1$*.)*

Problem 7 *Let triangle* ABC *have side lengths* $AB = 10$*,* $BC = 24$*, and* $AC = 26$*. Let* I *be the incenter of* ABC*. If the maximum possible distance between* I *and a point on the circumcircle of* ABC *can be expressed as* $a + \sqrt{b}$ *for integers* a *and* b *with* b *squarefree, find* $a + b$.

(The incenter *of any triangle* XYZ *is the intersection of the angle bisectors of* $\angle YXZ, \angle XZY$*, and* $\angle ZYX$*.)*

Problem 8 *How many terms in the expansion of*

$$(1 + x + x^2 + x^3 + \cdots + x^{2021})(1 + x^2 + x^4 + x^6 + \cdots + x^{4042})$$

have coefficients equal to 1011*?*