Mathathon Round

Girls in Math at Yale

February 27, 2021

Problem 1 *If* $5x + 3y - z = 4$, $x = y$, *and* $z = 4$, *find* $x + y + z$.

Problem 2 *How many ways are there to pick three distinct vertices of a regular hexagon such that the triangle with those three points as its vertices shares exactly one side with the hexagon?*

Problem 3 *Sirena picks five distinct positive primes,* $p_1 < p_2 < p_3 < p_4 < p_5$, *and finds that they sum to* 192. *If the product* $p_1p_2p_3p_4p_5$ *is as large as possible, what is* $p_1 - p_2 + p_3 - p_4 + p_5$*?*

Problem 4 *Suppose that* $\overline{A2021B}$ *is a six-digit integer divisible by* 9. *Find the maximum possible value of* $A \cdot B$.

Problem 5 *In an arbitrary triangle, two distinct segments are drawn from each vertex to the opposite side. What is the minimum possible number of intersection points between these segments?*

Problem 6 *Suppose that* a *and* b *are positive integers such that* $\frac{a}{b-20}$ *and* $\frac{b+21}{a}$ *are positive integers. Find the maximum possible value of* $a + b$.

Problem 7 *Peggy picks three positive integers between* 1 *and* 25, *inclusive, and tells us the following information about those numbers:*

- *Exactly one of them is a multiple of* 2;
- *Exactly one of them is a multiple of* 3;
- *Exactly one of them is a multiple of* 5;
- *Exactly one of them is a multiple of* 7;
- *Exactly one of them is a multiple of* 11.

What is the maximum possible sum of the integers that Peggy picked?

Problem 8 *What is the largest positive integer* k *such that* 2^k *divides* $2^{4^8} + 8^{2^4} + 4^{8^2}$*?*

Problem 9 *Find the smallest integer* n *such that* n *is the sum of* 7 *consecutive positive integers and the sum of* 12 *consecutive positive integers.*

Problem 10 *Prair picks a three-digit palindrome* n *at random. If the probability that* $2n$ *is also a palindrome can be expressed as* $\frac{p}{q}$ *in simplest terms, find* $p + q$. *(A palindrome is a number that reads the same forwards as backwards; for example,* 161 *and* 2992 *are palindromes, but* 342 *is not.)*

Problem 11 *If two distinct integers are picked randomly between* 1 *and* 50 *inclusive, the probability that their sum is divisible by* 7 *can be expressed as* $\frac{p}{q}$*, where* p *and* q *are relatively prime positive integers. Find* $p+q$*.*

Problem 12 *Ali is playing a game involving rolling standard, fair six-sided dice. She calls two consecutive die rolls such that the first is less than the second a "rocket." If, however, she ever rolls two consecutive die rolls such that the second is less than the first, the game stops. If the probability that Ali gets five rockets is* $\frac{p}{q}$*, where* p *and* q *are relatively prime positive integers, find* $p+q$*.*

Problem 13 *The triangle with vertices* $(0,0),(a,b)$*, and* $(a,-b)$ *has area* 10*. Find the sum of all possible positive integer values of* a*, given that* b *is a positive integer.*

Problem 14 *Elsa is venturing into the unknown. She stands on* $(0,0)$ *in the coordinate plane, and each second, she moves to one of the four lattice points nearest her, chosen at random and with equal probability. If she ever moves to a lattice point she has stood on before, she has ventured back into the known, and thus stops venturing into the unknown from then on. After four seconds have passed, the probability that Elsa is still venturing into the unknown can be expressed as* $\frac{a}{b}$ *in simplest terms. Find* $a+b$*.*

(A lattice point is a point with integer coordinates.)

Problem 15 *Let* $ABCD$ *be a square with side length* 4*. Points* X,Y*, and* Z*, distinct from points* A,B,C*, and* D*, are selected on sides* AD,AB*, and* CD*, respectively, such that* $XY=3, XZ=4$*, and* $\angle YXZ=90^\circ$*. If* $AX=\frac{a}{b}$ *in simplest terms, then find* $a+b$*.*

Problem 16 *Suppose trapezoid* $JANE$ *is inscribed in a circle of radius* 25 *such that the center of the circle lies inside the trapezoid. If the two bases of* $JANE$ *have side lengths* 14 *and* 30 *and the average of the lengths of the two legs is* $\sqrt{m}$*, what is* m*?*

Problem 17 *What is the radius of the circle tangent to the* x*-axis, the line* $y=\sqrt{3}x$*, and the circle* $(x-10\sqrt{3})^2+(y-10)^2=25$*?*

Problem 18 *Find the smallest positive integer* n *such that* $3n^3-9n^2+5n-15$ *is divisible by* 121 *but not* 2*.*