GiM 2024 Team Round Solutions

Yale Math Competitions

February 2024

1. To get from HLH13 to HLH17, Tim the Turtle can either walk 500 feet north and then 1200 feet west, or descend x feet into a tunnel, walk the distance between the two buildings in a straight line, and then ascend x feet. The two paths turn out to take the same time. If Tim descends at half the speed as compared to walking, how deep below ground is the tunnel, in feet?

   Proposed by: Stephen Xia

   Answer: $\boxed{100}$

   Solution: If Tim doesn't use the tunnel, he must walk $500 + 1200 = 1700$. If he uses the tunnel, he ascends/descends $2x$ feet and then walks the hypotenuse of a right triangle with legs $500, 1200$, so he walks a distance of $\sqrt{500^2 + 1200^2} = 1300$. However, he ascends/descends twice as slow, so he effectively has to travel a distance of $4x$ feet if he were traveling at a normal speed. Therefore we want $1300 + 4x = 1700$, so $x = \boxed{100}$.

2. Jane is jogging down a street at 5 miles per hour, leaving a trail of breadcrumbs. She begins next to the Hudson River and ends 10 miles away. Tina the Speedy Turtle begins at the same point, and runs at 20 miles per hour whenever there are breadcrumbs, and only 15 miles per hour without. How long of a head-start, in minutes, should Jane get so that Tina reaches the destination 40 minutes ahead of Jane?

   Proposed by: Stephen Xia

   Answer: $\boxed{45}$

   Solution: Let x be the time, in hours, that Jane gets as a headstart, and let t be the time after Tina starts that Jane and Tina meet. Then, at the point they meet, Jane has traveled for $x + t$ hours and Tina has traveled for t hours, and they must have traveled the same distance, so $5(x + t) = 20t$, so $x = 3t$. Now, Jane finishes the race in $\frac{10}{5} = 2$ hours, so Tina needs to finish the race in $\frac{4}{3} - x$ hours. She spends $t = \frac{x}{3}$ of this time running at 20 miles per hour, and after passing Jane, she runs at 15 miles per hour for the remaining $\frac{4}{3} - x - \frac{x}{3}$ hours. This means that $20\left(\frac{x}{3}\right) + 15\left(\frac{4}{3} - \frac{4x}{3}\right) = 10$, so $\frac{20x}{3} + 20 - 20x = 10$, or $x = \frac{3}{4}$. Therefore, Jane needs a $\boxed{45}$ minute headstart.

3. Gim, a variant of Nim, is played by two players. On each player's turn, they flip a fair coin. On heads, they add 1 to the commutative sum, and on tails, they add two. If the sum exceeds 6 on their turn, they win. The probability that the first player wins is $\frac{n}{m}$ in simplest form. Find $n + m$.

Proposed by: Stephen Xia

Answer: $\boxed{97}$

Solution: Let $f(x)$ be the probability that the first player wins when the counter is currently x. Then $f(6) = 1$, since they are guaranteed to exceed 6 after the coin flip. Furthermore, $f(5) = \frac{1}{2}$, since either the first player flips heads and sends the counter to 6, when they are guaranteed to lose, or to 7, where they win. On each turn, a player has a equal chance to go from x to $x+1$ or $x+2$ at which point the other person plays, so $f(x) = 1-\frac{1}{2}(f(x+1)+f(x+2))$. We can then compute backwards to get $f(4) = \frac{1}{4}, f(3) = \frac{5}{8}, f(2) = \frac{9}{16}, f(1) = \frac{13}{32}, f(0) = \frac{33}{64}$, so the answer is $33 + 64 = \boxed{97}$.

4. Define $f(x) = 2 + \dfrac{(3x)!(4x)!}{13^{13}} + 5^x + x^6$. What is the smallest positive integer x such that 13 divides $f(x)$?

Proposed by: Howard Dai

Answer: $\boxed{34}$

Solution: Note that all terms are guaranteed to be integers, except $\frac{(3x)!(4x)!}{13^{13}}$. To make this an integer, we must have at least 13 powers of 13 in the numerator. Note that for $x < 13$, we have no powers of 13 in the numerator, and for $13 \leq x < 26$, we have 7 powers of 13 in the numerator (think about how many multiples of 13 are contained within each factorial). For $x \geq 26$, we have at least 14 powers of 13 in the numerator. So, $x \geq 26$ and the term $\frac{(3x)!(4x)!}{13^{13}}$ will be divisible by 13. So, we can look at terms starting with 26. We now have:

$$2 + 5^x + x^6 \equiv 0 \bmod 13$$

$$5^x + x^6 \equiv -2 \bmod 13$$

Note that 5^x cycles in fours mod 13, as follows:

$$5^1 \equiv 5 \bmod 13$$

$$5^2 \equiv -1 \bmod 13$$

$$5^3 \equiv -5 \bmod 13$$

$$5^4 \equiv 1 \bmod 13$$

In addition, note that $x^{12} \equiv 1 \bmod 13$, so x^6 can only be -1 or $1 \bmod 13$ (or $0 \bmod 13$, if x is divisible by 13). Then clearly we must have $x^6 \equiv -1 \bmod 13$, $5^x \equiv -1 \bmod 13$. We can then proceed by testing values of 5^k for $k \equiv 2 \bmod 4$. We have $5^{26} \equiv -1 \bmod 13$, but $26^6 \equiv 0 \bmod 13$. $5^{30} \equiv -1 \bmod 13$, but $30^6 \equiv 4^6 \equiv 3^3 \equiv 1 \bmod 13$. $5^{34} \equiv -1 \bmod 13$, and we have $34^6 \equiv 8^6 \equiv (-5)^6 \equiv (-1)^3 \equiv -1 \bmod 13$, so 34 is the first number which satisfies $5^{34} \equiv -1 \bmod 13$ and $34^6 \equiv -1 \bmod 13$, and our answer is $\boxed{34}$.

5. In Brick Roll, a 2 by 1 by 1 rectangular prism starts with a square face on the ground. A red dot is placed at the center of the top face. Define a "roll" as a rotation over one of the brick's edges, where the pivot edge does not slip and the brick lands on a new face. The "direction" of a roll corresponds to the edge which it is pushed over (i.e. the direction towards which it

moves). After the brick is rolled north, east, and north (in that order), the total distance the red dot travels along its trajectory can be represented as a fraction in the form $\frac{a+\sqrt{b}+\sqrt{c}}{d}\pi$ such that b and c are not perfect squares and $\gcd(a,d)=1$. Find $a+b+c+d$.

Proposed by: Grant Zhang

Answer: $\boxed{24}$

Solution: Every time the brick rolls, it travels a quarter circle. We want to know the distance between the red dot and the pivot edge of the rotation. On the first roll, the dot is at a height 2 and horizontal distance $\frac{1}{2}$ from the pivot edge, so the rotation trajectory has radius $\frac{\sqrt{17}}{2}$, and a quarter circle then has arc length $\frac{\sqrt{17}}{4}\pi$. When it rolls horizontally, the dot is height $\frac{1}{2}$ above the ground and horizontal distance $\frac{1}{2}$ from the pivot edge, so the rotation trajectory has radius $\frac{\sqrt{2}}{2}$, and a quarter circle has arc length $\frac{\sqrt{2}}{4}\pi$. Finally, on the last roll, the dot is directly above the pivot edge at height $\frac{1}{2}$, so that is the radius. A quarter circle then has arc length $\frac{\pi}{4}$. Then the total trajectory length is $\frac{\sqrt{17}+\sqrt{2}+1}{4}\pi$, so the answer is $17+2+1+4=\boxed{24}$.

6. A $3\times3\times3$ cube is divided into 27 unit cubes. Remove 11 of these cubes such that the remaining cubes form one connected component (two cubes are only connected if they share a face). What is the maximum possible surface area remaining?

Proposed by: Stephen Xia

Answer: $\boxed{66}$

Solution: Our goal is to place 16 unit cubes within the $3\times3\times3$ constraint. Suppose we did not have this constraint - then the maximum surface area we could obtain is $\boxed{66}$, by placing all the unit cubes in a line. This value is achievable with the following configuration (top layer, middle layer, bottom layer, in order):

$$\begin{pmatrix}1&1&1\\0&0&1\\1&1&1\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&1&1\\1&0&1\\1&0&1\end{pmatrix}$$

Note that this is not the only valid configuration.

7. Find the number of sequences $a_1, a_2, \ldots, a_{20}$ such that $a_1=10$, $a_{20}=37$, and

$$|a_n-a_{n-1}| = 2-\frac{\left|a_{n-1}-\frac{37}{2}\right|}{\frac{37}{2}-a_{n-1}}$$

Proposed by: Neil He and Grant Zhang

Answer: $\boxed{153}$

Solution: We note that when $a_{n-1}<\frac{37}{2}$, the step size from a_{n-1} to a_n is 1 but when $a_{n-1}\geq\frac{37}{2}$, the step size is 3. We must reach a distance of 27 in 19 steps, and if we only step forward, we can reach 27 in 15 steps ($10\to11\to12\to13\to14\to15\to16\to17\to18\to19\to22\to25\to28\to31\to34\to37$), which implies we can afford to backtrack at most twice. In particular, if we backtrack on 19, we need 3 steps to recover

$(19 \to 16 \to 17 \to 18 \to 19)$, which uses all of our steps). Therefore, this case only has 1 possibility. Then, we count the number of ways to backtrack twice out of our 19 steps, which is $\binom{19}{2} = 171$. However, we have overcounted the number of ways we can backtrack on 19, which is invalid if we want to backtrack twice. There are two possibilities: either we backtracked before 19 and then once at 19, backtracked once at 19 and then backtracked after 19, or we backtracked twice at 19 (which is overcounted). The respective number of possibilities by PIE is $\binom{8}{1} + \binom{11}{1} - 1 = 18$. If we subtract this from the total we get the desired answer of $\boxed{153}$.

8. There are 10 cakes that Cindy and Katherine will divide them amongst themselves in 10 rounds. In the ith round, Cindy cuts the ith cake into two pieces, and Katherine then either chooses to take or pass. If she chooses to take, she first picks a piece for herself, and Cindy gets the other. If she chooses to pass, Cindy first picks a piece, and then Katherine gets the other. Katherine must pass one round during the game. If both people play optimally, maximizing the amount of cake they get, the number of cakes Katherine ends up with is $\frac{m}{n}$ in simplest form. Find $m + n$.

   Proposed by: Stephen Xia

   Answer: $\boxed{6143}$

   Solution: Observe that if Katherine has already passed, Cindy can only obtain $\frac{1}{2}$ of each cake - since if she cuts it unevenly, Katherine will simply always pick the largest piece. Let x_i be the most cake Katherine can get if there are i cakes available and she hasn't passed. If $i = 1$, Katherine must pass, so Cindy can give herself the entire cake, so $x_i = 0$. When $i = 2$, if Cindy cuts the cake into portions $c, 1 - c$ where $c > \frac{1}{2}$, then Katherine can either pass, so she gets $1 - c + \frac{1}{2}$, or take, so she gets c. She will always take the higher of these quantities, so Cindy wants to ensure they are equal (since one increases and the other decreases with respect to c). This occurs when $c = \frac{3}{4}$, so $x_2 = \frac{3}{4}$. In particular, we can repeat this logic, on the ith cake, if Cindy cuts in proportions $c, 1 - c$, Katherine can either take and get $c + x_{i-1}$, or pass and get $1 - c + \frac{i-1}{2}$, and setting them equal gives $c = \frac{i+1}{4} - \frac{x_{i-1}}{2}$. In particular, by plugging in and testing we find that Katherine can secure $\frac{n}{2} - \frac{1}{2^n}$ cakes, and for $n = 10$ this value is $5 - \frac{1}{1024} = \frac{5119}{1024}$ which gives an answer of $\boxed{6143}$.

9. Let S be the set of all polynomials of the form $(n + 1)x^n + nx^{n-1} + \ldots + 2x + 1$ for some positive integer n, and let R be the set of all complex roots of polynomials in S. Let a and b be real numbers such that $a \le |r| \le b$ for all elements r in R. The minimum possible value of $b - a$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

   Proposed by: Patrick Lu

   Answer: $\boxed{}$

   Solution: Note that

$$(n + 1)x^n + nx^{n-1} + \ldots + 2x + 1 = \sum_{i=1}^{n} \sum_{k=1}^{i}$$

10. 5 red marbles and 5 blue marbles are randomly arranged to form two adjacent circles, such

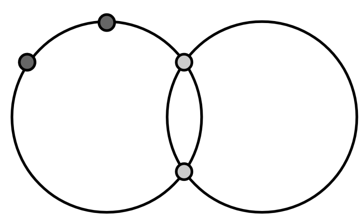
that each circle has 6 marbles (and 2 marbles are shared by each circle). An example configuration is shown to the right. Let a "move" be any rotation of one of the circles, such that each marble in the circle is moved by one spot. Let B represent the number of blue marbles in the righthand circle (including the shared marbles). Barbara wants to maximize B, and conducts two moves after seeing the initial random arrangement. Assuming she acts optimally, the expected value of B after Barbara makes her moves is $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

Proposed by: Howard Dai

Answer: $\boxed{103}$

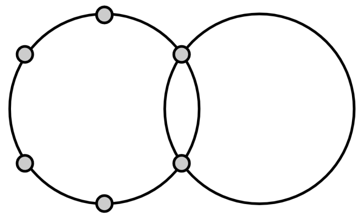
Solution: For simplicity, let the number of blue marbles in the righthand circle be Barbara's "score". By linearity of expectation, we can count her expected initial score X, and her expected increase in score Y after making moves (number of additional blue marbles Barbara is able to add to the right side). $X = 3$ because each marble in the right circle has a $\frac{1}{2}$ of initially being blue. Then it suffices to find the expected value of Y.

Clearly, Barbara can add at most 2 marbles, and at least 0. We can first count the number of cases in which Barbara can increase her score by 2. Because rotating the righthand circle does not change her score, the only case would be rotating the left circle twice. Then the only case where Barbara increases her score by 2 is when we have two red marbles in the middle area, and two consecutive blue marbles immediately adjacent as follows:

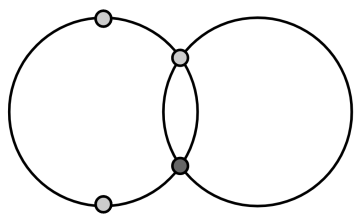


We have a symmetrical case where the blue marbles are on the bottom, and we double count the case where both the bottom and top are all occupied by blue marbles. For this case, we have 3 blue marbles and 3 red marbles remaining to place freely among the 6 remaining positions, giving us $\binom{6}{3}$ choices, and in the "double" case, we have 1 blue and 3 red, giving us $\binom{4}{1}$ overcounts. Then there are $2\binom{6}{3} - \binom{4}{1} = 36$ arrangements where Barbara can add 2 to her score.

Now we count the number of cases where Barbara cannot add anything to her score. First suppose the middle two marbles are both red. Then any blue marbles in the left circle within two rotations would allow Barbara to increase her score. Thus, we must have:

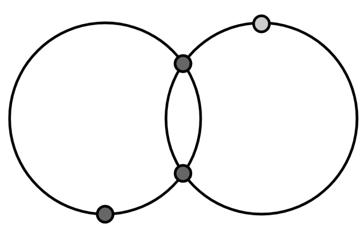


However, this is impossible, as we only have 5 red marbles. Now consider the case where one of the middle marbles is red, and one is blue. WLOG, suppose the top one is red and the bottom is blue. Note that if a blue marble is on the left circle right under the middle, a counterclockwise rotation increases her score. So, this must be red. Similarly, if a blue marble is on the left circle right above the middle, a counterclockwise rotation of the right circle and a clockwise rotation of the left circle increases her score. So, this must also be red. Then we have:



You can experiment with combinations of other marbles and rotations to see that there is no way to increase Barbara's score (because the most immediate marbles are both red). So, any arrangement falling under this case cannot be increased. There are 2 remaining red marbles and 4 remaining red marbles, so we have $2\binom{6}{2} = 30$ arrangements for this case (because there are also two symmetrical, if red is on top vs. bottom).

Finally, consider the case where both of the middle marbles are blue. Then the only way Barbara could increase her score would be a right rotation moving a red marble into the middle, and a left rotation "swapping" it out. This requires us to have some arrangement as follows:



We have a symmetric case flipped over the horizontal axis. However, we are overcounting the case where both symmetrical cases are satisfied. For the original case, we have 2 blue and 4 red remaining, so we have $\binom{6}{2}$ choices, and in the overcounted case, we have 1 blue and 3 red remaining, so we have $\binom{4}{1}$ choices. Overall, there are $2\binom{6}{2} - \binom{4}{1} = 26$ arrangements where Barbara can increase her score given two blue marbles in the middle, and there are $\binom{8}{3} = 56$ total arrangemenmts with two blue marbles, so we have $56 - 26 = 30$ arrangements with two blue marbles in the middle where Barbara cannot increase her score. Then in total, there are $30 + 30 = 60$ arrangements where Barbara cannot increase her score.

Overall, we have 36 arrangements where Barbara can increase her score by 2, and 60 arrangements where Barbara cannot increase her score. There are $\binom{10}{5} = 252$ total arrangements of marbles, giving us $252 - 60 - 36 = 156$ arrangements where Barbara increases her score by exactly 1. So, the expected increase in her score Y is $\frac{2\cdot 36 + 1\cdot 156}{252} = \frac{19}{21}$. Then the total expected number of marbles on her side is $3 + \frac{19}{21} = \frac{82}{21}$. This gives us $82 + 21 = \boxed{103}$.

11. The area of the hexagon in the complex plane whose vertices are the roots of the polynomial

$$ix^6 + \frac{37}{64}\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)x^3 + \frac{27}{64}$$

can be expressed as $\frac{a\sqrt{b}}{c}$ where a, b, c are integers, a and c are relatively prime, and b is not divisible by the square of any prime. Find $a + b + c$.

Proposed by: Stephen Xia

Answer: $\boxed{20}$

Solution: The motivation for this problem is to observe that the $\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)$ term is equivalent to $e^{i\frac{3\pi}{4}}$, which suggests a rotation. In particular, we can multiply the polynomial by -1 and factor it to get

$$((e^{i\frac{\pi}{4}}x)^3 - 1)\left((e^{i\frac{\pi}{4}}x)^3 + \left(\frac{3}{4}\right)^3\right)$$

Rotating the roots by $\frac{\pi}{4}$ does not affect the area of the hexagon, so we ignore the $e^{i\frac{\pi}{4}}$ coefficient. This implies that the roots of are the points $e^0, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$ and $\frac{3}{4}e^{i\frac{\pi}{3}}, \frac{3}{4}e^{i\pi}, \frac{3}{4}e^{i\frac{5\pi}{3}}$. On the complex plane, these are the vertices of two equilateral triangles, one of side length $\sqrt{3}$ and one of side length $\frac{3\sqrt{3}}{4}$ that are superimposed on each other. The area of the larger triangle is $\frac{3\sqrt{3}}{4}$, and the hexagon is formed by extending each side with a triangle of base $\sqrt{3}$ and height $\frac{1}{4}$, which contributes a total extra area of $\frac{3\sqrt{3}}{8}$, so the total is $\frac{9\sqrt{3}}{8}$ and the desired answer is $9 + 3 + 8 = \boxed{20}$.

12. There is a 1000×1000 grid of cats of distinct sizes. Each cat feels safe if they are adjacent to exactly one cat that's bigger than it. What is the maximum number of cats that feel safe? (Cats are only adjacent if they are horizontally or vertically adjacent.

Proposed by: Neil He

Answer: $\boxed{667332}$

Solution: Notice that it is a universal law that about 2/3 of the cats feel safe.
First note that if we view cats as vertices of a graph and where the edges are whether an adjacent cat is the bigger one, a scenario where all cats here safe is a spanning tree. The question thus becomes if we starting removing cats from the grid, what is the maximum number of cats we can have left so that the remaining graph is a spanning tree. Suppose we have c cats left. Note that removing a cat from the interior of the grid removes 4 edges. Also, we have to remove at least 1 cat from the boundary because otherwise the graph would contain a cycle. So we remove at least $4(1000^2 - c) - 1$ edges. Now note that since we want the remaining graph to be a spanning tree, it must have at least $c - 1$ edges. Thus we have $c-1 \geq 2000^2 - 4c + 1$ since there are 2000(999) edges in the first place. So we have $c \leq 667332$. Now we show a construction that works. Let n be the number of a row, then we follow these steps

(a) If $n = 1000$, remove the first cat.

(b) If $n \equiv 1 \pmod 6$, remove the 999-th cat.

(c) If $n \equiv 0, 2 \pmod 6$, remove the cats that are $k \pmod 6$ where $k \geq 2$ is even.

(d) If $n \equiv 3, 5 \pmod 5$, remove the cats that are $k \pmod 6$ where $k \geq 3$ is odd.

(e) if $n \equiv 4 \pmod 6$ and $n \neq 1000$, remove the second cat.

Note (a), (b) and (d) remove 1 cat each. (b) and (c) removes 499 cats. So we remove a total of $1 + 167 + 499 \cdot 333 + 499 \cdot 333 + 166 = 332668$ cat which leaves 667332 of them. Now we can check that each cat is next to exactly a cat that is larger than it. One can easily check that for each node (i, j) in row i and column j is either a leaf node or it is connected to exactly 2 cats. Then we can pick a direction of the edges such that each node is either a leaf or there is exactly incoming edge and exactly one out going edges. So the answer is 667332.