GiM 2024 Tiebreaker Round Solutions

Yale Math Competitions

February 2024

1. In triangle ABC, $AB = 20$ and $AC = 23$. Let M be the midpoint of BC, and let A' be the reflection of A over M. Let D and E be the feet of the perpendiculars from A' to AB and AC, respectively. The circumcircle of $\triangle MDE$ meets BC again at a point $X \neq M$. If $MX = 3$, the length BC can be represented as $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m + n$.

   Proposed by: Patrick Lu

   Answer: $\boxed{45}$

   Solution: 43/2

2. Collatz has a function

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ 5x + 1 & \text{if } x \text{ is odd} \end{cases}$$

   Define $f_n(x)$ where $f_1(x) = f(x)$ and $f_n(x) = f(f_{n-1}(x))$ when $n \geq 2$. What is $f_{2024}\left(\sum_{k=0}^{30} 2^{4k+1} + 2^{4k}\right)$?

   Proposed by: Darwin Deng

   Answer: $\boxed{3}$

   Solution: The initial quantity is odd, since it has a +1 term at the end. Therefore,

$$\begin{aligned} f_{2024}\left(\sum_{k=0}^{30} 2^{4k+1} + 2^{4k}\right) &= f_{2023}\left(1 + \sum_{k=0}^{30} (2^2 + 1)(2^{4k+1} + 2^{4k})\right) \\ &= f_{2023}\left(1 + \sum_{k=0}^{30} 2^{4k+3} + 2^{4k+2} + 2^{4k+1} + 2^{4k}\right) \\ &= f_{2023}(2^{124}) \end{aligned}$$

   so our desired quantity can be reduced to $f_{1899}(1)$. From here, we can see that the cycle should repeat $1, 6, 3, 16, 8, 4, 2, 1, \cdots$. In particular, since $f_{1899}(1)$, and the cycle repeats every 7 times, and $1899 \equiv 2 \pmod 7$, then $f_{1999}(1) = f_2(1) = \boxed{3}$.

3. Jacqueline has three (not necessarily distinct) polynomials $p(x)$, $q(x)$, and $r(x)$ such that each polynomial is one of $x + 1$, $x + 2$, or $x + 3$. For each combination of $p(x)$, $q(x)$, and

$r(x)$, Jacqueline finds the product $p(x)q(x)r(x)$. Jacqueline then sums all of the products (which are, again, not necessarily distinct) to obtain the polynomial $w(x)$. Find the sum of the coefficients of $w(x)$.

Proposed by: Grant Zhang

Answer: $\boxed{729}$

Solution: By symmetry, as $p(x)$ rotates from $(x+1),(x+2),(x+3)$, the set of combinations of $q(x)r(x)$ are the same, so we can factor the expression as $(x+1+x+2+x+3)\left(\sum q(x)r(x)\right)$. However, we can repeat this logic, so the total sum should be $(x+1+x+2+x+3)^3$, which simplifies to $(3x+6)^3$. To get the sum of coefficients, we plug in $x=1$, which gives $\boxed{729}$.