GiM 2024 Individual Round Solutions

Yale Math Competitions

February 2024

1. Trains run from New York to New Haven, departing every half-hour beginning at 6:00 AM. Trains run from New Haven to Boston, departing every 45 minutes beginning at 6:15 AM. The distance between New Haven and New York is 75 miles and the distance between New Haven and Boston is 200 miles. Trains can travel at 50 miles per hour, and only these two trains are running today. Jessica decides to depart from New York at 6:00 AM. What is the shortest amount of time it would take her to get to Boston by train, in minutes?

   Proposed by: Stephen Xia

   Answer: $\boxed{345}$

   Solution: Jessica departs on the New York-New Haven train at 6:00 AM, and travels 75 miles to New York, which takes her $\frac{75}{50} = 1.5$ hours. Therefore, she arrives at New Haven at 7:30 AM. However, trains going from New Haven to Boston depart at 6:15 AM, 7:00 AM, 7:45 AM, etc. Therefore, the soonest she can depart from New Haven is 7:45 AM, at which point she travels for $\frac{200}{50} = 4$ hours, so she arrives in Boston at 11:45 AM. Therefore, she has spent 5 hours and 45 minutes total, which is $\boxed{345}$ minutes.

2. Cat and Claire are playing a crossword game. Cat says: "I know the digits of 2-Across are all the same, and 3-Down is a perfect square whose digits are in strictly ascending order." Claire, who knows the number in 1-Down, says "The number in 1-Down is divisible by 13, and I know how to complete the grid." What is 1-Down?

   Proposed by: Stephen Xia

   Answer: $\boxed{91}$

   Solution: Since 3-Down is a 3-digit perfect square with strictly ascending digits, it must be either $13^2 = 169, 16^2 = 256$, or $17^2 = 289$. However, since all digits of 2-Across are the same, the units digit of 1-Down must be the same as the hundreds digit of 3-Down. Claire declares she knows the grid completely, meaning there can only be one option for her: if her units digit was 2, she wouldn't be able to know if 3-Down is 256 or 289. Therefore, the units digit of 1-Down is 1, and the only multiple of 13 satisfying this is $\boxed{91}$.

3. Five 1s and four 0s are distributed in a 3×3 grid at random. The probability there exists a row, column, or diagonal with all three numbers being the same can be expressed as $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.
   Proposed by: Stephen Xia

Answer: $\boxed{118}$

Solution: We will count the number of ways to arrange the grid so that there is no three-in-a-row. Every row must have at least one 0, otherwise a row will have three 1s. Therefore, one row must have two 0s and the other two rows will have one 0. We casework based on the row with two 0s.

We will reference this grid for locations: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Case 1: The top or bottom row has two 0s (this is symmetric, so we will just consider row 123 and multiply by 2 at the end).
Subcase 1: The two zeros are in $1, 2$ (this is symmetric to $2, 3$, so we will multiply by 2). Then, we must put the last two zeros in $6, 7$, which gives 1 option. This case contributes $2 \cdot 1 = 2$.
Subcase 2: The two zeros are in $1, 3$. Then, the two remaining zeros could be in $(4, 8), (5, 8), (6, 8)$, so this case contributes 3.
The total then is $(2 + 3) \cdot 2 = 10$.
Case 2: The middle row has two 0s.
Subcase 1: The two zeros are in $4, 5$ (this is symmetric to $5, 6$, so we will multiply by 2). Then, we could place the last two zeros in $(2, 9), (3, 8), (3, 9)$, so this case contributes $3 \cdot 2 = 6$.
Subcase 2: The two zeros are in $4, 6$. This case contributes 0, since we cannot place the remaining two zeros.
The total is then 6.
Then, we have 16 arrangements. There are a total of $\binom{9}{4} = 126$ arrangements, and 110 of them produce a three-in-a-row, so the desired probability is $\frac{110}{126} = \frac{55}{63}$, so the answer is $\boxed{118}$.

4. Ten equally spaced points $A, B, C, D, E, F, G, H, I, J$ are drawn on the circumference of a unit circle. What is $AB^2 + AC^2 + AD^2 + \ldots + AJ^2$?

Proposed by: Howard Dai

Answer: $\boxed{20}$

Solution: Note that $\overline{AF}$ is a diameter of the unit circle, and thus $AF = 2$. Because $\overline{AF}$ is a diameter, any triangle containing $\overline{AF}$ and a third point on the circumference of the circle will be a right triangle. For example, $\triangle ABF$ is a right triangle with hypotenuse $\overline{AF}$. By symmetry, observe that $BF = AE$ (by reflection), so we have $AE^2 + AB^2 = AF^2 = 4$ by Pythagorean Theorem. By a similar argument, we have $AC^2 + AD^2 = 4, AJ^2 + AG^2 = 4, AI^2 + AH^2 = 4$. So, by pairing up our sides in this manner, we have a total of $4 \cdot 4 + 4 = \boxed{20}$.

5. What is $\sum_{k=1}^{12} \sqrt{(k-1)k(k+1)(k+2)+1}$?

Proposed by: Stephen Xia

Answer: $\boxed{716}$

Solution: We have that $(k-1)(k)(k+1)(k+2) = k^4 + 2k^3 - k^2 - 2k$, and we can factor

$k^4 + 2k^3 - k^2 - 2k + 1 = (k^2 + k - 1)^2$, so we want to compute $\sum_{k=1}^{n}(k^2 + k - 1) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - n = \frac{(2n^3 + 3n^2 + n) + 3(n^2 + n) - (6n + 6)}{6} = \frac{2n^3 + 6n^2 - 2n - 6}{6}$, which for $n = 12$ evaluates to 716.

6. A *subsequence* of a word is formed by taking some characters of the word in order.

   For example, ATTIC is a subsequence of the word MATHEMATICS, formed by the underlined characters.

   Compute the number of distinct 4 character subsequences of the word MATHEMATICS.

   Proposed by: Nikhil Kalghatgi

   Answer: $\boxed{300}$

   Solution: There are 11 letters in the word MATHEMATICS. The repeated letters among these are M, A, and T, each of which are repeated twice. Therefore, there are $\binom{11}{4} = 330$ choices of 4 characters, but some of these choices lead to the same subsequence.

   In particular, subsequences that include one repeated letter and no letters in between the different occurrences of that repeated letter can be formed in two ways. Namely, by choosing either the first or the second occurence of that repeated letter.

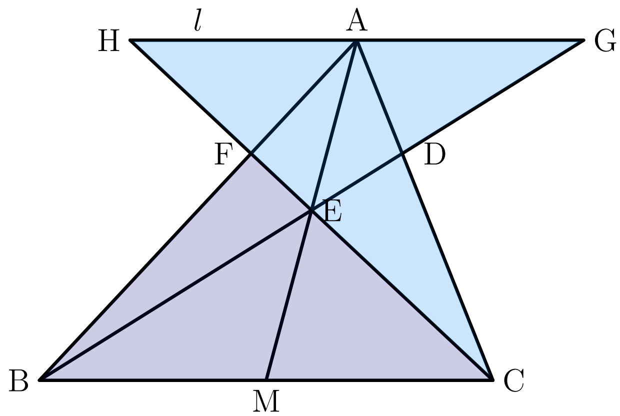
   For each choice of repeated letter (M, A, or T) there are 5 characters that do not appear between occurrences of that repeated letter, and therefore $\binom{5}{3} = 10$ subsequences that are repeated twice. Therefore in total, there are $3 \cdot 10 = 30$ subsequences that appear twice, and this leaves $330 - 30 = \boxed{300}$ distinct subsequences.

7. In acute triangle ABC, M is the midpoint of BC, and point D located on segment AC. Let BD and AM intersect at E. Let CE and AB intersect at F. Call the line parallel to BC passing through A line ℓ. Then, let ℓ intersect BE at G and ℓ intersect CE at H, and we have $[BCF] = [CHD] + [HDG]$. The value of $\frac{[AFD]}{[BEC]}$ when written in simplest form is $\frac{m}{n}$. Find $m + n$.

   Proposed by: Michael Gao

   Answer: $\boxed{11}$

   Solution:



First, by Ceva's Theorem, we must have that $AD \cdot MC \cdot BF = DC \cdot MB \cdot FA$, and since $MB = MC$, we have $\frac{AF}{FB} = \frac{AD}{DC}$, this implies $AFD \sim ABC$, so $FD \parallel BC$. Then, suppose the ratio of $AD : DC = k : 1 - k$ and let $BC = 1$, and WLOG let the area of $[ABC] = 1$. Then by area ratios (since ABF, BFC share the same height their area ratio is dependent only on the ratio of their bases) we have $[ABF] = k, [BFC] = 1 - k$. Since $AHF \sim FCB$ and $\frac{AF}{FB} = \frac{k}{1-k}$, the area $[AHF] = \left(\frac{k}{1-k}\right)^2 \cdot (1-k) = \frac{k^2}{1-k}$, and the area of $[ADG]$ can be computed in the same way. Therefore, we have $\frac{2k^2}{1-k} + k = 1 - k$, which implies $k = \frac{1}{3}$. That means that the area $[AFD] = \frac{1}{3^2} \cdot 1 = \frac{1}{9}$. Then, since $EFD \sim ECB$, and $\frac{FD}{BC} = k$, then $\frac{FE}{EC} = k$, so $\frac{[BEF]}{[BEC]} = \frac{FE}{EC} = k$, so $[BEC] = \frac{1}{k+1} \cdot (1-k) = \frac{1}{2}$. Therefore the desired ratio is $\frac{\frac{1}{9}}{\frac{1}{2}} = \frac{2}{9}$ and the answer is $2 + 9 = \boxed{11}$.

8. Find the number of ordered tuples $(k_1, \ldots, k_n)$ satisfying $\displaystyle\sum_{i=1}^{n} \frac{1}{k_i} = 1$ and $\displaystyle\prod_{i=1}^{n} k_i \leq 2024$.

Proposed by: Neil He

Answer: $\boxed{227}$

Solution: We will casework based on the size of the tuple. Observe that if there are 5 items in the tuple, then the smallest possible product of the tuples is obtained from $\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$, which gives $5^5 > 2024$ which is invalid.
Within each subcase, we check for valid tuples in ascending order of the smallest number.
Case 1: We have (1) is the only tuple.
Case 2: We have $(2, 2)$ is the only tuple.
Case 3: We have $(3, 3, 3), (2, 3, 6), (2, 4, 4)$, which gives $1 + 6 + 3 = 10$ total cases.
Case 4: We have $(2, 5, 5, 10), (4, 4, 4, 4), (3, 4, 4, 6), (2, 3, 8, 24), (2, 3, 10, 15), (2, 6, 6, 6), (2, 3, 7, 42), (2, 4, 6, 12), (2, 4, 8, 8), (3, 3, 4, 12), (3, 3, 6, 6), (2, 3, 9, 18), (2, 4, 5, 20), (2, 3, 12, 12)$ which gives $12 + 1 + 12 + 24 + 24 + 4 + 24 + 24 + 12 + 12 + 6 + 24 + 24 + 12 = 215$ cases.
Summing over the total gives $\boxed{227}$.

9. In triangle ABC, let D be the point on segment BC such that $\angle BAD = \angle ACB$. The circle through A tangent to BC at D intersects AB at $E \neq A$ and AC at $F \neq A$. Let lines EF and

BC intersect at a point P. If $AE = 11$, $DE = 10$, and $EF = 20$, compute the perimeter of triangle PDF.

Proposed by: Jeffrey Xu

Answer: $\boxed{231}$

Solution: Call the circle ω. Because BD is tangent to ω at D, we have $\angle EDB = \angle EAD = \angle ACB = \angle EFD$. Therefore, ED must be parallel to AC, so $AEDF$ must be an isosceles trapezoid. Thus, $DF = AE = 11$.

Because $ED \parallel AC$, we must have $\angle EDF = \angle DFC$. Because $\angle EFD = \angle DCF$, we know that triangles EDF and DFC are similar. Thus, we have $\frac{DC}{DF} = \frac{EF}{ED} \implies DC = 22$ and $\frac{FC}{DF} = \frac{DF}{ED} \implies FC = \frac{121}{10}$.
Triangles PED and PFC must also be similar (as $ED \parallel FC$), so we must therefore have

$$\begin{aligned}\frac{PE + PD}{PF + PC} &= \frac{ED}{FC}\\ \frac{PE + PD}{PE + PD + 42} &= \frac{100}{121}\\ 21\left(PE + PD\right) &= 4200\\ PE + PD &= 200\end{aligned}$$

Thus, the perimeter of triangle PDF is $(PE + PD) + EF + FD = 200 + 20 + 11 = \boxed{231}$.

10. Find the greatest positive integer n such that $(n + 1)!$ divides

$$\left(13231^n - 1\right)\left(13231^{n-1} - 1\right)\dots\left(13231 - 1\right).$$

Proposed by: Patrick Lu

Answer: $\boxed{99}$

Solution: Note that $13231 = 101 \cdot 131$. If $n \geq 100$, then $101 \mid (n + 1)!$ but

$$\left(13231^n - 1\right)\left(13231^{n-1} - 1\right)\dots\left(13231 - 1\right) \equiv \pm 1 \pmod{101}.$$

Thus $n \leq 99$. We show all $n \leq 99$ have the desired property.
Consider any prime p that divides $(n + 1)!$. We have $p \leq n + 1 \leq 100$. Note that

$$\nu_p\left((n+1)!\right) = \left\lfloor \frac{n+1}{p} \right\rfloor + \left\lfloor \frac{n+1}{p^2} \right\rfloor + \dots + \left\lfloor \frac{n+1}{p^{\lfloor \log_p (n+1) \rfloor}} \right\rfloor < \frac{n+1}{p} + \frac{n+1}{p^2} + \dots = \frac{n+1}{p-1}.$$

Since $\nu_p\left((n+1)!\right)$ is an integer, $\nu_p\left((n+1)!\right) \leq \left\lfloor \frac{n+1}{p-1} \right\rfloor$ By Fermat's little theorem, p divides $13231^{k(p-1)} - 1$ for all positive integers k. Thus p divides all of

$$13231^{p-1} - 1, 13231^{2(p-1)} - 1, \dots, 13231^{\lfloor \frac{n}{p-1} \rfloor (p-1)} - 1,$$

implying that ν_p of the given product is greater than or equal to $\left\lfloor \frac{n}{p-1} \right\rfloor$. If $p-1 \nmid n+1$, then $\left\lfloor \frac{n}{p-1} \right\rfloor = \left\lfloor \frac{n+1}{p-1} \right\rfloor$. If $p-1 \mid n+1$, then since $\nu_p\left((n+1)!\right) < \frac{n+1}{p-1}$, $\nu_p\left((n+1)!\right) \le \left\lfloor \frac{n+1}{p-1} \right\rfloor - 1 = \left\lfloor \frac{n}{p-1} \right\rfloor$, as desired.

11. For integers $k > 1$ define $f(k)$ as the sum of all numbers of the form $\frac{1}{n}$ such that $\frac{1}{n}$ terminates when written in base k. For example, in base 10, $\frac{1}{2} = 0.5$ terminates but $\frac{1}{3} = 0.333...$ does not terminate, so $\frac{1}{2}$ would be part of the sum representing $f(10)$ but $\frac{1}{3}$ would not. If $\sum_{k|10!, k>1} f(k)$ can be expressed as $\frac{m}{n}$ such that m, n are relatively prime positive integers, find $m + n$.

Proposed by: Vismay Sharan

Answer: $\boxed{10841}$

Solution: In order for a fraction to terminate in base k, its denominator must be composed of only the prime factors of k. (So for base 10, every terminating decimal can be written as a fraction whose denominator is $2^a 5^b$ for some $a, b \ge 0$.) Thus, suppose $k = p_1^{e_1} p_2^{e_2} \cdots p_n^{e^n}$. Then, n must be of the form $p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n}$ for $x_i \ge 0$. In particular, we can factorize

$$\sum \frac{1}{n} = \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \cdots\right)\left(1 + \frac{1}{p_2} + \frac{1}{p_2^2} + \cdots\right)\cdots\left(1 + \frac{1}{p_n} + \frac{1}{p_n^2} + \cdots\right)$$

which simplifies to $\prod_{i=1}^{n}\left(\frac{p_i}{p_i - 1}\right)$. We can prime factorize $10! = 2^8 \times 3^4 \times 5^2 \times 7$. In particular, the only primes that can divide k are $2, 3, 5, 7$. For example, if k is divisible by only 2, $f(k) = 2$, if k is divisible by 2 and 3, $f(k) = 3$, if k is divisible by $2, 3, 5$, $f(k) = 2 \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{4}$.

Now we must find a way to sum over all divisors of 10!. Each divisor can be uniquely represented by individual choices of exponents for each prime. For example, we have 9 choices for powers of 2, 8 of which are nonzero, requiring us to multiply by $(1 + \frac{1}{2} + \dots) = 2$ and 1 of which is zero, so we multiply by 1. Similarly, we have 4 nonzero choices for powers of 3 (requiring us to multiply by $\frac{3}{2}$), 2 nonzero choices for powers of 5, and 1 nonzero choice for powers of 7. **Note that we must subtract by 1 to ignore the $f(1)$ case. We can then represent the sum of all possible combinations by:

$$(1+2+2+2+2+2+2+2+2)(1+\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+\frac{3}{2})(1+\frac{5}{4}+\frac{5}{4})(1+\frac{7}{6}) - 1$$

$$= (1+8(2))(1+4(\frac{3}{2})(1+2(\frac{5}{4}))(1+\frac{7}{6}) - 1$$

$$= (17)(7)(\frac{7}{2})(\frac{13}{6})$$

$$= \frac{10817}{12}$$

So, our sum is $10817 + 12 = \boxed{10829}$.

12. Find the minimum possible value of $n-m$, where n, m are real numbers such that $m \leq \frac{4+b^2}{b} + \frac{4+a^2}{a} \leq n$ for all positive real numbers a, b that satisfy $\frac{a^2+1}{a} + \frac{b^2+1}{b} \leq 18$ and $a^2b + b^2a \leq 3$.

Proposed by: Neil He

Answer: $\boxed{46}$

Solution: This is equivalent to find the tightest bounds on $\frac{4+b^2}{b} + \frac{4+a^2}{a}$. Let $x = \frac{1}{a+b}$ and $y = \frac{1}{ab}$. Then we have $\frac{a^2+1}{a} + \frac{b^2+1}{b} \leq 18 \implies y \leq 18x - 1$ and $a^2b + b^2a \leq 3 \implies y \leq \frac{1}{3x}$. Now from AM-GM we also have that $y \geq 4x^2$. Now note that we are trying to the bounds of $\frac{4+b^2}{b} + \frac{4+a^2}{a} = n$. This becomes $\frac{4y+1}{x} = n \implies y = \frac{nx}{4} - \frac{1}{4}$. Note that the bounds of n is determined by the slope of the line from (x, y) to the point $(0, -\frac{1}{4})$. By plotting out the 3 inequalities we found earlier, we see that the largest slopes comes from the intersection of $y = 18x - 1$ and $y = \frac{1}{3x}$. This intersection is at the point $(\frac{1}{6}, 2)$ (we take the positive solution because a, b are positive real numbers). The slope is $\frac{27}{2}$. For the lower bound, we can see it's the the slope from the point tangent to the parabola $y = 4x^2$. Of course we can easily the slope at that point is 2 by taking the derivative. Without calculus, notice that this is the solution to the equation $mx - \frac{1}{4} = 4x^2$ with $m^2 - 4 = 0 \implies m = 2$ (note that m has to be positive again because of the positivity of a, b). So the bounds are $(8, 54)$. So the answer is $54 - 8 = 46$.