MMATHS 2024 Individual Round Solutions

Yale Math Competitions

October 2024

1. Let $ab^2 = 126, bc^2 = 14, cd^2 = 128, da^2 = 12$. Find $\frac{bd}{ac}$.

   Proposed by: Benjamin Wu

   Answer: $\boxed{96}$

   Solution:
   $$(ab^2 \cdot cd^2)/(bc^2 \cdot da^2) = (bd/ac) = (126 \cdot 128)/(14 \cdot 12) = 96$$

2. Grant has a box with 6 red balls, 5 blue balls, 4 green balls, 3 yellow balls, 2 orange balls, and 1 purple ball. Grant selects 6 balls at random, without replacement. Let P be the probability that Grant selects six balls of different colors, and let Q be the probability that Grant selects six balls of the same color. What is $\frac{P}{Q}$?

   Proposed by: Howard Dai

   Answer: $\boxed{720}$

   Solution: There are $6! = 720$ cases for having one ball of each color, and 1 case for all same (the only way is if we have 6 red balls).

3. Let $f(x)$ be a function, where if q is an integer, then $f(\frac{1}{q}) = q$, and if m and n are real numbers, $f(m \cdot n) = f(m) \cdot f(n)$. If $f(\sqrt{2})$ can be written as $\frac{\sqrt{a}}{b}$ where a is prime, then what is $a + b$?

   Proposed by: Noah Ripke

   Answer: $\boxed{4}$

   Solution: We first notice that $f(\frac{1}{1}) = f(1) = 1$ and $f(2) = f(\sqrt{2}\sqrt{2}) = f(\sqrt{2})^2$. So we would like to find $f(2)$. We notice that $f(2 \cdot \frac{1}{2}) = f(1) = f(2) \cdot f(\frac{1}{2}) = 1$. But we know that $f(\frac{1}{2}) = 2$ so $f(2) = \frac{1}{2}$ (In general, we can show that $f(r) = \frac{1}{r}$ for rational numbers r). Thus $f(\sqrt{2}) = \frac{\sqrt{2}}{2}$ so $a + b = 2 + 2 = 4$.

4. Let ABC be an equilateral triangle with side length 1. Then, let M be the midpoint of $\overline{BC}$. The area of all points within ABC that are closer to M than either A, B, or C can be expressed as the fraction $\frac{\sqrt{a}}{b}$ where a is not divisible by the square of any prime and b is a positive integer. Find $a + b$.

   Proposed by: Stephen Xia

Answer: $\boxed{11}$

Solution: Draw the perpendicular bisectors of segments AM, BM, CM. All described points are within the rectangle formed by these perpendicular bisectors. We have a rectangle with side lengths $1/2$, $\sqrt{3}/4$, giving us an area of $\frac{\sqrt{3}}{8}$.

5. Two subsets are called *disjoint* if they do not share any common elements. Compute the number of ordered tuples (A, B, C), where A, B, and C are subsets (not necessarily distinct or non-empty) of $\{1, 2, 3, 4, 5\}$ such that A and B are disjoint and B and C are disjoint.

   Proposed by: Owen Zhang

   Answer: $\boxed{3125}$

   Solution: Consider a specific element in $\{1, 2, 3, 4, 5\}$ and the 8 possibilities of which of $\{A, B, C\}$ it is in. The only three that are invalid are: in $\{A, B\}$, in $\{B, C\}$, and in $\{A, B, C\}$. In particular, any element can be only in A, only in B, only in C, in none of the three sets, or only in A and C. Hence, over all 5 elements there are $5^5 = 3125$ possibilities.

6. How many unique 7 digit numbers satisfy the following?

   - All digits are distinct digits from 1 - 7.
   - The first digit (from the left) is divisible by 1.
   - The two-digit number formed by the first two digits is divisible by 2.
   - The three-digit number formed by the first three digits is divisible by 3.
   - The four-digit number formed by the first four digits is divisible by 4.
   - The five-digit number formed by the first five digits is divisible by 5.
   - The six-digit number formed by the first six digits is divisible by 6.

   Proposed by: Yushu Zhang, Maya Viswanathan

   Answer: $\boxed{4}$

   Solution: Since the number formed by the first 5 digits is divisible by 5, the 5th digit must be 5. Also, since the first six digits are divisible by 6, they are divisible by 3. Their sum is 28 minus the last digit, which must be a multiple of 3. This means the last digit is either 1, 4, or 7. Also, since there are 3 even numbers among 1-7 (2, 4, 6), and the numbers made by the first 2, 4, and 6 digits must be even, we must use the numbers 2, 4, and 6 in the 2nd, 4th, and 6th, positions, though not necessarily in that order. Then, the last digit must be either 1 or 7. We can divide into two cases.

   Case 1: The last digit is 1. Then the first and third digits are 3 and 7, in some order. The sum of the first three digits is the second digit $+3 + 7$, and it must be divisible by three. The only possibility for the second digit that works, out of $2, 4$ and 6, is 2. Then since the first four digits are divisible by 4, the third and fourth digits are divisible by 4. Since the third digit is odd and the fourth digit cannot be 2, the fourth digit must be 6. Then the sixth digit is 4. The only digits that have not yet been decided are the first and third which could either be 7 or 3. Both possibilities are options, so there are two options in this case.

Case 2: The last digit is 7. Then the first and third digits are 1 and 3, in some order. The sum of the first three digits is the second digit $+3+1$, and it must be divisible by three. The only possibility for the second digit that works, out of 2, 4, and 6, is 2. Then since the first four digits are divisible by 4, the third and fourth digits are divisible by 4. Since the third digit is odd and the fourth digit cannot be 2, the fourth digit must be 6. Then the sixth digit is 4. The only digits that have not yet been decided are the first and third which could either be 1 or 3. Both possibilities are options, so there are two options in this case.

So, there are four options in total.

7. The sum $\sum_{x=-5}^{5}\sum_{y=-5}^{5}\frac{2^x3^y}{(1+2^x)(1+3^y)}$ can be expressed as a fraction $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

   Proposed by: Aaron Xiong

   Answer: $\boxed{125}$

   Solution: Split the two sums up. Considering only the x sum, let the quantity equal S. Then notice that replacing x with $-x$ keeps the sum the same. Adding these two sums and cancelling gives that $2S = 11$, so $S = 11/2$. Similar logic for y gives $11/2$ as well, so multiplying gives $121/4$.

   In fact, the base of the exponent doesn't even matter for such sums as long as the bounds are symmetric.

8. Let circle A have radius 9, and let circle B have radius 5 and be internally tangent to circle A. The largest radius r such that there are two circles with radius r that lie inside circle A, are externally tangent to each other, and externally tangent with Circle B can be expressed as a fraction $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

   Proposed by: Grant Zhang

   Answer: $\boxed{229}$

   Solution: We will use the centers of each of the four circles and connect each center. Let the two circles with radius r be called O_1 and O_2. Call the tangent point between O_1 and O_2 to be P.

   We note that when O_1 and O_2 have the largest possible radius r, they will also be internally tangent to circle A. Then, note that $AO_1 = 9-r$, $O_1O_2 = 2r$ (and hence $O_1P = r$), and $BO_1 = r+5$.

   We then know that ΔAPO_1 is a right triangle, so $AP = \sqrt{(9-r)^2 - r^2} = \sqrt{81-18r}$. Then, note that ΔBPO_1 is also a right triangle, so $BP = \sqrt{(r+5)^2-(r)^2} = \sqrt{10r+25}$, but also note that $BP = BA + AP = 4 + \sqrt{81-18r}$. Therefore, it suffices to solve $\sqrt{10r+25} = 4+\sqrt{81-18r}$. After performing some algebra, we obtain $r = 0, \frac{180}{49}$, and as we want the largest radius r, we hence have $r = \frac{180}{49}$, which gives us answer of $180+49 = 229$.

9. 2048 frogs are sitting in a circle and each have a \$1 bill. After each minute, each frog will independently give away each of their \$1 bills to either the closest frog to their left or the closest frog to their right with equal probability. If a frog has \$0 at the end of any given minute, then they will not give any money but may receive money. The expected number of

frogs to have at least \$1 after 3 minutes can be denoted as a common fraction in the form $\frac{a}{b}$. Find $a+b$.

Proposed by: Grant Zhang

Answer: $\boxed{2873}$

Solution: Note that since every dollar must be transferred after each minute, then note that each person can only end up receiving one of 4 different \$1 bills: the ones originally owned by the person 3 seats to the left, the person 1 seat to the left, the person 1 seat to the right, and the person 3 seats to the right.

The probability that a dollar bill from 1 seat away eventually goes to you is $\frac{1}{8}$, while the probability that a dollar bill from 3 seats away goes to you is $\frac{3}{8}$. So, overall, the probability that at least one dollar bill goes to you is:

$$1-(\frac{7}{8})^2(\frac{5}{8})^2=\frac{2871}{4096}$$

Then the expected number of frogs with at least one dollar bill is $\frac{2871}{4096}\cdot 2048=\frac{2871}{2}$.

10. Find the sum of all prime numbers n such that $\binom{20242024n}{n}\equiv 2024 \pmod{n}$.

Proposed by: Stephen Xia

Answer: $\boxed{41}$

Solution:$\binom{20242024n}{n}\equiv\binom{20242024}{1} \pmod{n}$ when n is prime by Wilson's theorem, so we want n to divide $20242024-2024=2024000$, which means $n=2,5,11,23$, which gives an answer of 41.

11. Let n be the least possible value of

$$\sqrt{x^2+y^2-2x+6y+19}+\sqrt{x^2+y^2+8x-4y+21}$$

Find n^2.

Proposed by: Vismay Sharan

Answer: $\boxed{66}$

Solution: We can first factor to get:

$$\sqrt{(x-1)^2+(y+3)^2+9}+\sqrt{(x+4)^2+(y-2)^2+1}$$

Note that minimizing this expression is the same as minimizing the following expression, with the constraint that $z=0$:

$$\sqrt{(x-1)^2+(y+3)^2+(z-3)^2}+\sqrt{(x+4)^2+(y-2)^2+(z+1)^2}$$

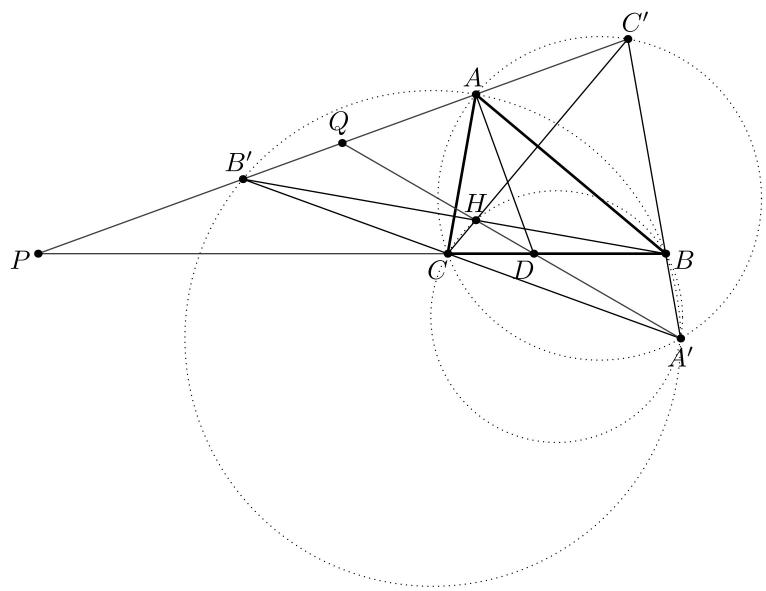
Then minimizing this expression is equivalent to minimizing the sum of distances from a point to $(1,-3,3)$ and $(-4,2,-1)$, with the constraint that the point must lie on the z axis. Then clearly, the minimal sum is achieved by placing this point on the segment between the two points, and the sum of the distances is the same as the distance between $(1,-3,3)$ and $(-4,2,-1)$, which is $\sqrt{66}$.

12. Let ABC be a triangle with $\angle A = 60^\circ$ and orthocenter H. Let B' be the reflection of B over AC, C' be the reflection of C over AB, and A' be the intersection of BC' and $B'C$. Let D be the intersection of $A'H$ and BC. If $BC = 5$ and $A'D = 4$, then the area of $\triangle ABC$ can be expressed as $a\sqrt{b} + \sqrt{c}$, where a, b, and c are positive integers, and b and c are not divisible by the square of any prime. Find $a + b + c$.

Proposed by: Patrick Lu

Answer: $\boxed{42}$

Solution:



First we show that AD bisects $\angle A$. The angle condition implies that B', A, C' are collinear. Let $P = BC \cap B'C'$ and $Q = A'H \cap B'C'$. Since $B'C'$ is the exterior angle bisector of $\angle BAC$, we have

$$\frac{AB}{BP} = \frac{AC}{PC}.$$

Note that B, H, B' are collinear and C, H, C' are collinear. Then by the theorems of Ceva and Menelaus,

$$-1 = (B', C'; P, Q) \stackrel{A'}{=} (C, B; P, D) = \frac{CP}{BP} \div \frac{CD}{BD}.$$

Therefore

$$\frac{AC}{PC} \cdot \frac{BP}{AB} \cdot \frac{CP}{BP} \div \frac{CD}{BD} = -1,$$
$$\frac{AC}{AB} = \frac{CD}{BD},$$

implying AD bisects $\angle CAB$ by the angle bisector theorem.

Now we show that $\angle DAA' = \angle AA'D$, implying that $AD = A'D$. By orthocenter reflections, $\angle CAB + \angle CHB = 180^\circ$. Since

$$\angle CAB = \angle C'A'B' = \angle BA'C',$$

we know quadrilateral $A'BHC$ is cyclic. Since

$$\angle A'B'A = \angle CB'A = \angle C'BA,$$

we know quadrilateral $ABA'B'$ is cyclic. Similarly, $ACA'C'$ is cyclic. Then

$$\angle C'A'A = \angle C'CA = \angle HCA = \angle ABH = \angle ABB' = \angle BB'A = \angle BA'A,$$

implying that $A'A$ bisects $\angle C'A'B'$.

Thus

$$\begin{aligned} \angle DAA' &= \angle DAB - \angle A'AB = \angle AA'B' - \angle A'B'B \\ &= \angle AA'B' - \angle HBC = \angle AA'B' - \angle DA'C = \angle AA'D, \end{aligned}$$

and $AD = A'D = 4$.

Now let $AB = x$, $AC = y$. By the angle bisector theorem, $BD = \frac{5x}{x+y}$ and $CD = \frac{5y}{x+y}$. Applying Stewart's theorem on $\triangle ABC$ with Cevian AD, we have

$$\begin{gathered} x^2\left(\frac{5y}{x+y}\right) + y^2\left(\frac{5x}{x+y}\right) = 80 + 5\left(\frac{5x}{x+y}\right)\left(\frac{5y}{x+y}\right), \\ (x+y)^2xy = 16(x+y)^2 + 25xy. \end{gathered}$$

By the law of cosines on $\triangle ABC$, we have

$$\begin{gathered} 25 = x^2 + y^2 - xy \\ (x+y)^2 = x^2 + 2xy + y^2 = 3xy + 25. \end{gathered}$$

Then

$$\begin{gathered} (3xy+25)xy = 16(3xy+25) + 25xy \\ 3(xy)^2 - 48xy - 400 = 0 \\ xy = \frac{24 + 4\sqrt{111}}{3}. \end{gathered}$$

Thus

$$\begin{aligned} [\triangle ABC] &= \frac{1}{2}xy\sin(60^\circ) \\ &= \left(\frac{\sqrt{3}}{4}\right)\left(\frac{24+4\sqrt{111}}{3}\right) \\ &= 2\sqrt{3} + \sqrt{37}, \end{aligned}$$

and $a + b + c = \boxed{42.}$