Math Majors of America Tournament for High Schools

2019 Mixer Test Solutions

1. An ant starts at the top vertex of a triangular pyramid (tetrahedron). Each day, the ant randomly chooses an adjacent vertex to move to. What is the probability that it is back at the top vertex after three days?

Answer: $\frac{2}{9}$

Solution: The only way the ant can get back to the top vertex after 3 days is if it doesn't go to the top vertex on the second day and then does go back to the top vertex on the third day. (The first day's choice is irrelevant by symmetry.) So the desired probability is $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$.

2. A square "rolls" inside a circle of area π in the obvious way. That is, when the square has one corner on the circumference of the circle, it is rotated clockwise around that corner until a new corner touches the circumference, then it is rotated around that corner, and so on. The square goes all the way around the circle and returns to its starting position after rotating exactly $720°$. What is the area of the square?

Answer: $2 - \sqrt{3}$

Solution: If the square simply spins around the circle, it rotates $-360°$, so in order to achieve $720° = -360° + 12(90°)$ it must touch the circle 12 times. In other words, the points where some corner of the square touches the circle (of radius 1) are the vertices of a regular dodecagon. Thus, the side length of the dodecahedron (which is also the side length of the square) is $2\sin(15°) = \frac{\sqrt{3}-2}{\sqrt{2}}$. Squaring this quantity gives that our desired area is $2 - \sqrt{3}$.

3. How many ways are there to fill a 3×3 grid with the integers 1 through 9 such that every row is increasing left-to-right and every column is increasing top-to-bottom?

Answer: 42

Solution: The fastest way to solve this problem is by case work. Note that the top left corner must have 1 and the bottom right corner must have 9. The 2 must be in either the top middle box or the middle left box; we consider only the first of these two cases and double our answer at the end. Now, consider the value of the middle right box, which can take the values $8, 7, 6$ (since the 5 boxes above and to the left of it must have smaller values). These possibilities give, respectively, $5(2), 4(2), 3(2) - 1$ ways to fill in the rest of the grid. Summing and doubling gives $2(10 + 8 + 3) = 42$ ways. This problem can also be solved with the hook length formula for standard Young tableaux.

4. Noah has an old-style M&M machine. Each time he puts a coin into the machine, he is equally likely to get 1 M&M or 2 M&M's. He continues putting coins into the machine and collecting M&M's until he has at least 6 M&M's. What is the probability that he actually ends up with 7 M&M's?

Answer: $\frac{21}{64}$

Solution: Let a_n be the probability that n is "skipped" in the sequence of M&M partial sums. This question is asking for the value of a_6. Clearly, $a_0 = 0$. An integer $n > 0$ can be skipped only if $n - 1$ was achieved and then the next coin got 2 M&M's, which happens with probability $\frac{1}{2}$. So $a_n = \frac{1}{2}(1 - a_{n-1})$. We can now easily compute $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{3}{8}$, $a_4 = \frac{5}{16}$, $a_5 = \frac{11}{32}$, $a_6 = \frac{21}{64}$. Alternatively, one can use generating functions or the theory of linear recurrences to obtain the explicit formula $a_n = \frac{1}{3}(1 - (-\frac{1}{2})^n)$ and then plug in $n = 6$.

5. Erik wants to divide the integers 1 through 6 into nonempty sets A and B such that no (nonempty) sum of elements in A is a multiple of 7 and no (nonempty) sum of elements in B is a multiple of 7. How many ways can he do this? (Interchanging A and B counts as a different solution.)

Answer: 6

Solution: If $x \in A$, then we must have $7 - x \in B$. Hence, each of the pairs $\{1, 6\}$, $\{2, 5\}$, and $\{3, 4\}$ contains an element of A and an element of B. There are $2^3 = 8$ ways to split these pairs between A and B. Now, we just need to check that the sum of the elements of

A is not a multiple of 7 (and the corresponding result holds for B because $1+2+3+4+5+6=21$ is a multiple of 7). By trial and error, we see that we need to avoid $A=\{1,2,4\}$ and $A=\{6,5,3\}$. This leaves us with $8-2=6$ possibilities.

6. A subset of $\{1,2,3,4,5,6,7,8\}$ of size 3 is called *special* if whenever a and b are in the set, the remainder when $a+b$ is divided by 8 is not in the set. (a and b can be the same.) How many special subsets exist?

Answer: 6

Solution: We can take any 3 odd numbers, which we can do in $\binom{4}{3}=4$ ways. Now, consider the case where our set contains at least one even number. Clearly, our even number cannot be 8 since $8+8\equiv 8 \pmod 8$. Note that we cannot have multiple even numbers: if one of these is 4, then $2+2\equiv 6+6\equiv 4 \pmod 8$ is bad; and if we have 2 and 6, then double any odd number is either 2 or 6 modulo 8, which is also bad. If our even number is 2, then we cannot have 1 or 5 (which both double to 2 modulo 8), but that leaves us with 3 and 7, which sum to 2 modulo 8, so our set cannot contain 2. By symmetry, we cannot have 6. If our set contains 4, then our odd numbers must differ by exactly 2. We cannot take $\{4,1,3\}$ (since $3+1\equiv 4 \pmod 8$) or $\{4,5,7\}$ (since $5+7\equiv 4 \pmod 8$), but $\{4,3,5\}$ and $\{4,7,1\}$ work. This gives a total of $4+2=6$ special sets.

7. Let $F_1=F_2=1$, and let $F_n=F_{n-1}+F_{n-2}$ for all $n\geq 3$. For each positive integer n, let $g(n)$ be the minimum possible value of

$$|a_1F_1+a_2F_2+\cdots+a_nF_n|,$$

where each a_i is either 1 or -1. Find $g(1)+g(2)+\cdots+g(100)$.

Answer: 34

Solution: First, $g(1)=1$. Next, $|1-1|=1$ gives $g(2)=0$, and $|2-1-1|=0$ gives $g(3)=0$. I claim that for $n\geq 4$, we have $g(n)\leq g(n-3)$: Choose $a_1,a_2,\ldots,a_n$ so that $a_1F_1+a_2F_2+\cdots+a_nF_n=g(n)$, then $|(F_{n+3}-F_{n+2}+F_{n+1})+(a_1F_1+a_2F_2+\cdots+a_nF_n)|=g(n)$, as desired. Thus, $g(n)\in\{0,1\}$ for all n (by induction). Note that the parity of $|a_1F_1+a_2F_2+\cdots+a_nF_n|$ does not depend on the a_i's, so we can conclude that $g(n+3)=g(n)$ for all $ngeq4$. So $g(n)=0$ unless $n\equiv \pmod 3$, in which case $g(n)=1$. This gives $g(1)+g(2)+\cdots+g(100)=34$.

8. Find the smallest positive integer n with base-10 representation $\overline{1a_1a_2\cdots a_k}$ such that $3n=\overline{a_1a_2\cdots a_k1}$.

Answer: 142857

Solution: Let $a=\overline{a_1a_2\cdots a_k}$, so that $3(10^k+a)=10a+1$ and $3\cdot 10^k-1=7a$. To minimize n, we need to minimize k. Note that $3\cdot 10^k\equiv 3^{k+1}\equiv 1 \pmod 7$. The smallest k that satisfies this congruence is $k=5$, so $a=\frac{3\cdot 10^5-1}{7}=42857$ and $n=142857$.

9. How many ways are there to tile a 4×6 grid with L-shaped triominoes? (A triomino consists of three connected 1×1 squares not all in a line.)

Answer: 18

Solution: If you break the 4×6 grid into 4 small 2×3 grids, we get $2^4=16$ possibilities. Otherwise, brute force shows that there are only 2 possibilities, for a total of $16+2=18$.

10. Three friends want to share five (identical) muffins so that each friend ends up with the same total amount of muffin. Nobody likes small pieces of muffin, so the friends cut up and distribute the muffins in such a way that they maximize the size of the smallest muffin piece. What is the size of this smallest piece?

Answer: $\frac{5}{12}$

Solution: Each friend ends up with $\frac{5}{3}$ muffins. We can divide the muffins as $\frac{1}{2}+\frac{1}{2}$, $\frac{5}{12}+\frac{7}{12}$, $\frac{5}{12}+\frac{7}{12}$, $\frac{5}{12}+\frac{7}{12}$, $\frac{5}{12}+\frac{7}{12}$ and distribute the pieces as $\frac{1}{2}+\frac{7}{12}+\frac{7}{12}=\frac{5}{3}$, $\frac{1}{2}+\frac{7}{12}+\frac{7}{12}=\frac{5}{3}$, $\frac{5}{12}+\frac{5}{12}+\frac{5}{12}=\frac{5}{3}$. This shows that it is possible to have the smallest piece have size $\frac{5}{12}$. We now show that this division is optimal. If any muffin is cut into more than 2 pieces, then one of those pieces has size at most $\frac{1}{3}$, and this is already worse than $\frac{5}{12}$, so we can assume that each muffin is cut into exactly 2 pieces. (We don't have to worry about whole muffins because we don't lose anything by dividing them into $\frac{1}{2}+\frac{1}{2}$.) Hence, there are 10 pieces of muffin, and some student gets at least $\lceil\frac{10}{3}\rceil=4$ pieces of muffin. Because these pieces sum to $\frac{5}{3}$, one of them must have size $\frac{5}{12}$ or smaller. This completes the proof.

Numerical tiebreaker problems:

11. S is a set of positive integers with the following properties:

(a) There are exactly 3 positive integers missing from S.

(b) If a and b are elements of S, then $a+b$ is an element of S. (We allow a and b to be the same.)

How many possibilities are there for the set S?

Answer: 4

Solution: Identify the possible sets of "holes". Clearly, 1 must be a hole. If 2 is also a hole, then the third hole could be 3, 4, or 5. It can't be bigger than that because 3, 4, and 5, are clearly sufficient to generate everything bigger than 6. If 2 is not a hole, then 3 must be a hole (since otherwise 2 and 3 would generate everything, in which case there is only a single hole). The third hole must be an odd number because 2 generates all the even numbers. In fact, it must be 5: if 5 is not a hole, then 5 and 2 generate all the integers bigger than 6. In total, there are 4 possibilities.

12. In the trapezoid $ABCD$, both $\angle B$ and $\angle C$ are right angles, and all four sides of the trapezoid are tangent to the same circle. If $\overline{AB} = 13$ and $\overline{CD} = 33$, find the area of $ABCD$.

Answer: 429

Solution: Let $x = \overline{BC}$ be the height of the trapezoid. Note that pairs of opposite sides must have the same sum of lengths, so the $\overline{DA}$ has length $46-x$. By the Pythagorean theorem, $x^2+400 = (46-x)^2$, so $x = \frac{429}{23}$ and the area of the trapezoid is $23x = 429$.

13. Alice wishes to walk from the point $(0,0)$ to the point $(6,4)$ in increments of $(1,0)$ and $(0,1)$, and Bob wishes to walk from the point $(0,1)$ to the point $(6,5)$ in increments of $(1,0)$ and $(0,1)$. How many ways are there for Alice and Bob to get to their destinations if their paths never pass through the same point (even at different times)?

Answer: 5292

Solution: It is easier to count the pairs of paths that do intersect. We claim that there is a bijection between the following sets (for general m and n):

$$\left\{\begin{array}{l}\text{pairs of intersecting}\\ \text{paths from } (0,0)\\ \text{to } (m,n) \text{ and from}\\ (0,1) \text{ to } (m,n+1)\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\text{pairs of paths from}\\ (0,0) \text{ to } (m,n+1)\\ \text{and from } (0,1) \text{ to}\\ (m,n)\end{array}\right\}.$$

The bijection is the following: suppose the paths A from $(0,0)$ to (m,n) and B from $(0,1)$ to $(m,n+1)$ have their first point of intersection at (x,y). Then we let $f(A,B) = (C,D)$, where C agrees with A "until" (x,y) and with B "after" (x,y), and where D agrees with B "until" (x,y) and with A "after" (x,y). To see that this map is a bijection, note that every pair of paths from $(0,0)$ to $(m,n+1)$ and from $(0,1)$ to (m,n) must be intersecting, so the same operation of swapping after the first intersection is the inverse map of f.

Finally, we apply complementary counting. There are

$$\binom{m+n}{m}^2$$

pairs of paths from $(0,0)$ to (m,n) and from $(0,1)$ to $(m,n+1)$, and there are

$$\binom{m+n+1}{m}\binom{m+n-1}{m}$$

pairs of paths from $(0,0)$ to $(m,n+1)$ and from $(0,1)$ to (m,n), so the number of pairs of non-intersecting paths from $(0,0)$ to (m,n) and from $(0,1)$ to $(m,n+1)$ is

$$\binom{m+n}{m}^2 - \binom{m+n+1}{m}\binom{m+n-1}{m}.$$

Plugging in $m=6$ and $n=4$ gives $210^2 - (462)(84) = 5292$.

14. The continuous function $f(x)$ satisfies $9f(x+y) = f(x)f(y)$ for all real numbers x and y. If $f(1) = 3$, what is $f(-3)$?

Answer: 243

Solution: Clearly, $f(x)$ is always nonnegative, and it can't have any real zeroes because then (by continuity arguments) we would have $f(0) = 0$. Hence, we can write $f(x) = 3^{g(x)}$ for some continuous function $g(x)$. Taking the base-3 logarithm of both sides of the functional equation gives $2 + g(2x) = 2g(x)$. It is well-known that solutions to this equation must be linear, so write $g(x) = ax + b$. Then $2 + (2ax + b) = 2(ax + b)$ implies $2 + b = 2b$ and $b = 2$. From $f(1) = 3$, we have $g(1) = 1 = a + 2$ and hence $a = -1$. So $g(x) = -x + 2$. In particular, $g(-3) = 2 + 3 = 5$ and $f(-3) = 3^5 = 243$.