Math Majors of America Tournament for High Schools

2019 Mixer Test

Name: __________

Team : __________

1. __________

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12. __________

13. __________

14. __________

– 75 minutes
– no calculators
– simplify answers

1. An ant starts at the top vertex of a triangular pyramid (tetrahedron). Each day, the ant randomly chooses an adjacent vertex to move to. What is the probability that it is back at the top vertex after three days?

2. A square "rolls" inside a circle of area π in the obvious way. That is, when the square has one corner on the circumference of the circle, it is rotated clockwise around that corner until a new corner touches the circumference, then it is rotated around that corner, and so on. The square goes all the way around the circle and returns to its starting position after rotating exactly $720°$. What is the area of the square?

3. How many ways are there to fill a 3×3 grid with the integers 1 through 9 such that every row is increasing left-to-right and every column is increasing top-to-bottom?

4. Noah has an old-style M&M machine. Each time he puts a coin into the machine, he is equally likely to get 1 M&M or 2 M&M's. He continues putting coins into the machine and collecting M&M's until he has at least 6 M&M's. What is the probability that he actually ends up with 7 M&M's?

5. Erik wants to divide the integers 1 through 6 into nonempty sets A and B such that no (nonempty) sum of elements in A is a multiple of 7 and no (nonempty) sum of elements in B is a multiple of 7. How many ways can he do this? (Interchanging A and B counts as a different solution.)

6. A subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ of size 3 is called *special* if whenever a and b are in the set, the remainder when $a + b$ is divided by 8 is not in the set. (a and b can be the same.) How many special subsets exist?

7. Let $F_1 = F_2 = 1$, and let $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. For each positive integer n, let $g(n)$ be the minimum possible value of

$$|a_1F_1 + a_2F_2 + \cdots + a_nF_n|,$$

where each a_i is either 1 or -1. Find $g(1) + g(2) + \cdots + g(100)$.

8. Find the smallest positive integer n with base-10 representation $\overline{1a_1a_2\cdots a_k}$ such that $3n = \overline{a_1a_2\cdots a_k1}$.

9. How many ways are there to tile a 4×6 grid with L-shaped triominoes? (A triomino consists of three connected 1×1 squares not all in a line.)

10. Three friends want to share five (identical) muffins so that each friend ends up with the same total amount of muffin. Nobody likes small pieces of muffin, so the friends cut up and distribute the muffins in such a way that they maximize the size of the smallest muffin piece. What is the size of this smallest piece?

Numerical tiebreaker problems:

11. S is a set of positive integers with the following properties:

(a) There are exactly 3 positive integers missing from S.

(b) If a and b are elements of S, then $a + b$ is an element of S. (We allow a and b to be the same.)

How many possibilities are there for the set S?

12. In the trapezoid $ABCD$, both $\angle B$ and $\angle C$ are right angles, and all four sides of the trapezoid are tangent to the same circle. If $\overline{AB} = 13$ and $\overline{CD} = 33$, find the area of $ABCD$.

13. Alice wishes to walk from the point $(0, 0)$ to the point $(6, 4)$ in increments of $(1, 0)$ and $(0, 1)$, and Bob wishes to walk from the point $(0, 1)$ to the point $(6, 5)$ in increments of $(1, 0)$ and $(0, 1)$. How many ways are there for Alice and Bob to get to their destinations if their paths never pass through the same point (even at different times)?

14. The continuous function $f(x)$ satisfies $9f(x + y) = f(x)f(y)$ for all real numbers x and y. If $f(1) = 3$, what is $f(-3)$?