. . . . Mathathon Round 1 (2 points each) . . . .

1. A small pizza costs \$4 and has 6 slices. A large pizza costs \$9 and has 14 slices. If the MMATHS organizers got at least 400 slices of pizza (having extra is okay) as cheaply as possible, how many large pizzas did they buy?

Answer: 26

Solution: Note that the least common multiple of 6 and 14 is 42, so there should be fewer than 42 slices coming from small pizzas. So we get the first 364 slices from 26 large pizzas. For the remaining 36 slices, it's cheapest to get 6 small pizzas (instead of 3 large pizzas, or 2 large pizzas and 2 small pizzas, or a large pizza and 4 small pizzas). So the answer is 26.

2. Rachel flips a fair coin until she gets a tails. What is the probability that she gets an even number of heads before the tails?

Answer: $\frac{2}{3}$

Solution: The probability that her first tails comes on the n-th flip is 2^{-n}, so the probability that she gets her first tails on an odd-numbered flip is $2^{-1} + 2^{-3} + 2^{-5} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$.

3. Find the unique positive integer n that satisfies $n! \cdot (n+1)! = (n+4)!$.

Answer: 6

Solution: Dividing both sides by $(n+1)!$ gives $n! = (n+1)(n+3)(n+4)$. Explicit computation shows that $n = 6$ works. (The right-hand side is larger for $n < 6$, and the left-hand side is larger for $n > 6$.)

. . . . Mathathon Round 2 (3 points each) . . . .

4. The Portland Malt Shoppe stocks 10 ice cream flavors and 8 mix-ins. A milkshake consists of exactly 1 flavor of ice cream and between 1 and 3 mix-ins. (Mix-ins can be repeated, the number of each mix-in matters, and the order of the mix-ins doesn't matter.) How many different milkshakes can be ordered?

Answer: 1640

Solution: If there is only 1 mix-in, there are are 8 options. If there are 2 of the same mix-in, there are again 8 options, and the same goes for 3 of the same mix-in. If there are 2 different mix-ins, there are $\binom{8}{2} = 28$ options. If there are 3 distinct mix-ins, there are $\binom{8}{3} = 56$ mix-ins. If there are 2 of one mix-in and a third different mix-in, there are $(8)(7) = 56$ options. In total, there are $8 + 8 + 8 + 28 + 56 + 56 = 164$ possibilities for mix-ins. Multiply this by the number of ice cream flavors to get 1640 total different milkshakes.

5. Find the minimum possible value of the expression $(x)^2 + (x+3)^4 + (x+4)^4 + (x+7)^2$, where x is a real number.

Answer: $\frac{197}{8}$

Solution: The expression is symmetric about $x = \frac{7}{2}$, so we want to set this equal to 0. Then the expression evaluates to $2[(\frac{7}{2})^2 + (\frac{1}{2})^4] = 2[\frac{49}{4} + \frac{1}{16}] = \frac{197}{8}$. Calculus also works but is messier.

6. Ralph has a cylinder with height 15 and volume $\frac{960}{\pi}$. What is the longest distance (staying on the surface) between two points of the cylinder?

Answer: $15 + \frac{16}{\pi}$

Solution: Let the circle have radius r. The area of the base of the cylinder is $\frac{960}{15\pi} = \frac{64}{\pi} = \pi r^2$, so $r = \frac{8}{\pi}$, and the circumference is 16. It is clear (by, say, approximating the cylinder by a convex polytope and taking limits) that any extremal configuration will have each point either in the middle of one of the bases, halfway up the side of the cylinder, or at a point where the side touches one of the bases. We can immediately see that the only potential candidates on this list are (1) where the points are the centers of the two bases, (2) where both points lie on the boundary, directly opposite each other, and (3) where both points lie halfway up the side of the cylinder, directly opposite each other. Direct computation gives that the shortest distance is $15 + \frac{16}{\pi}$ in case (1), $\sqrt{15^2 + 8^2} = 17$ in case (2), and 8 in case (3) The largest of these comes from case (1).

. . . . Mathathon Round 3 (4 points each) . . . .

7. If there are exactly 3 pairs (x, y) satisfying $x^2 + y^2 = 8$ and $x + y = (x - y)^2 + a$, what is the value of a?

Answer: -4

Solution: Consider $u = x + y$ and $v = x - y$, so that the two equations become $u^2 + v^2 = 16$ and $u = v^2 + a$. Note that the number of solutions (u, v) equals the number of solutions (x, y) (just a change of coordinates). The first equation defines a circle of radius 4 centered at the origin, and the second equation defines a parabola opening upwards with its minimum at the point $(0, a)$. Looking at graphs of these curves shows that we need $a = -4$. (Substituting for u in the first equation and using the quadratic formula also works.)

8. If n is an integer between 4 and 1000, what is the largest possible power of 2 that $n^4 - 13n^2 + 36$ could be divisible by? (Your answer should be this power of 2, not just the exponent.)

Answer: 2048

Solution: Factor $n^4 - 13n^2 + 36 = (n^2 - 4)(n^2 - 9) = (n - 2)(n + 2)(n - 3)(n + 3)$. Either the first 2 factors are even or the last 2 factors are even. So the best we can get is when either $n - 2$ or $n + 2$ is a power of 2, in which case the other contributes a factor of 4. So the answer is $512 \cdot 4 = 2048$.

9. Find the sum of all positive integers $n \geq 2$ for which the following statement is true: "for any arrangement of n points in three-dimensional space where the points are not all collinear, you can always find one of the points such that the $n - 1$ rays from this point through the other points are all distinct."

Answer: 14

Solution: The statement is vacuously true for $n = 2$. For $n = 3$ and $n = 4$, the statement is obvious. Consider $n = 5$. If no 3 of the points are collinear, then any point can "see" all of the others. If any 4 of the points are collinear, then the last point can see all of the other points. Finally, suppose some 3 points are collinear (but no 4 points are collinear). If the middle point can see all of the other points, then we are done. Otherwise, the fourth point is between this point and the fifth point, and this fourth point can see all of the other points. So the statement is true for $n = 5$. For $n \geq 6$, the statement is false because of the following configuration: put $n - 2$ points on a line, and make one of the points on the end be the midpoint of the segment connecting the last 2 points. Thus, the answer is $2 + 3 + 4 + 5 = 14$.

.... Mathathon Round 4 (6 points each) ....

10. Donald writes the number 12121213131415 on a piece of paper. How many ways can he rearrange these fourteen digits to make another number where the digit in every place value is different from what was there before?

Answer: 420

Solution: Since half of the digits are 1's, all of the 1's must get sent to the other digits. For the remaining digits, there are 7 choices for the location of the 5, 6 choices for the location of the 4, $\binom{5}{2} = 10$ choices for the locations of the 3's, and then only 1 choice for the locations of the 2's. This gives a total of 420.

11. A question on Joe's math test asked him to compute $\frac{a}{b} + \frac{3}{4}$, where a and b were both integers. Because he didn't know how to add fractions, he submitted $\frac{a+3}{b+4}$ as his answer. But it turns out that he was right for these particular values of a and b! What is the largest possible value that a could have been?

Answer: -3

Solution: $\frac{a+3}{b+4} = \frac{a}{b} + \frac{3}{4} = \frac{4a+3b}{4b}$ gives $(4b)(a + 3) = (b + 4)(4a + 3b)$. Then $0 = 3b^2 + 16a$, so $a = \frac{-3b^2}{16}$. Thus, a is negative and a multiple of 3, so the largest possibility is $a = -3$ (which corresponds to $b = 4$).

12. Christopher has a globe with radius r inches. He puts his finger on a point on the equator. He moves his finger 5π inches North, then π inches East, then 5π inches South, then 2π inches West. If he ended where he started, what is largest possible value of r?

Answer: 15

Solution: After moving North, he is at a latitude where the latitudinal circle has half the perimeter of the equatorial circle. In other words, if this latitude line is inclined by an angle θ, then we have $\cos(\theta) = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$. Then we know that $5\pi = (2\pi r)\frac{\pi/3}{2\pi}$, so $r = 15$. (We remark that smaller values of r arise from the possibility that Christopher moves his finger more than once all the way around when he is moving East.)

.... Mathathon Round 5 (7 points each) ....

13. Suppose $\triangle ABC$ is an isosceles triangle with $\overline{AB} = \overline{BC}$, and X is a point in the interior of $\triangle ABC$. If $m\angle ABC = 94°$, $m\angle ABX = 17°$, and $m\angle BAX = 13°$, then what is $m\angle BXC$ (in degrees)?

Answer: 77°

Solution: Construct the point Y inside of $\triangle ABC$ by creating a 17°-angle from $\overline{BC}$ at B and a 13°-angle from $\overline{BC}$ at C. Note that $m\angle XBY = 94° - 2(17°) = 60°$. And $\triangle XBY$ is isosceles, so it is in fact equilateral. Since $\angle XYC = 360° - 60° - 150° = 150°$, we see that $\triangle XYC$ is congruent to $\triangle BYC$. Thus, $m\angle BCX = m\angle CBX = 77°$.

14. Find the remainder when $\sum_{n=1}^{2019} 1 + 2n + 4n^2 + 8n^3$ is divided by 2019.

Answer: 1346

Solution: Compute $\sum_{n=1}^{2019} 1+2n+4n^2+8n^3 = (1)(2019)+(2)(\frac{(2019)(2020)}{2})+(4)(\frac{(2019)(2020)(4039)}{6})+(8)(\frac{(2019)^2(2020)^2}{4})$. The first, second, and fourth terms vanish modulo 2019, so we are left with $(\frac{(2)(2020)(4039)}{3})(2019)$. The numerator is $(2)(1)(1) = 2$ modulo 3, so the entire term is equivalent to $(\frac{2}{3})(2019) = 1346$ modulo 2019.

15. How many ways can you assign the integers 1 through 10 to the variables a, b, c, d, e, f, g, h, i, and j in some order such that $a < b < c < d < e$, $f < g < h < i$, $a < g$, $b < h$, $c < i$, $f < b$, $g < c$, and $h < d$?

Answer: 240

Solution: Since there are no restrictions on j, we can choose any of the 10 values for j and then find the relative order of the other nine variables. Now, consider only a through i. We know that a and f are smaller than everything else. We can have $a < f$ or $f < a$, for a total of 2 possibilities here. We know that b and g are the next two smallest elements, so we can have $b < g$ or $g < b$, for another 2 possibilities. Similarly, c and h are the next two smallest elements, so we once again get 2 possibilities. For the remaining variables, we can have $d < e < i$, $d < i < e$, or $i < d < e$, which gives 3 possibilities. In total, we have $(10)(2)(2)(2)(3) = 240$ possibilities. We remark that this problem is asking for the number of linear extensions of a poset.

.... Mathathon Round 6 (8 points each) ....

16. Call an integer n *equi-powerful* if n and n^2 leave the same remainder when divided by 1320. How many integers between 1 and 1320 (inclusive) are equi-powerful?

Answer: 16

Solution: An integer n is equipowerful (i.e., idempotent) modulo 1320 if and only if $n^2 - n = n(n-1) \equiv 0 \pmod{1320}$. Since n and $n-1$ are always relatively prime, this is equivalent to saying that each prime power in the prime factorization of 1320 divides either n or $n-1$, i.e., n is equivalent to either 0 or 1 modulo each prime power. Write $1320 = 2^3 * 3 * 5 * 11$. For each prime power, choose a residue of either 0 or 1, and for each quadruple there is one corresponding integer modulo 1320 by the Chinese Remainder Theorem. So the answer is $2^4 = 16$.

17. There exists a unique positive integer $j \leq 10$ and unique positive integers $n_j, n_{j+1}, \ldots, n_{10}$ such that

$$j \leq n_j < n_{j+1} < \cdots < n_{10}$$

and

$$\binom{n_{10}}{10} + \binom{n_9}{9} + \cdots + \binom{n_j}{j} = 2019.$$

Find $n_j + n_{j+1} + \ldots + n_{10}$.

Answer: 66

Solution: Try choosing the n_i's greedily starting with $i = 10$. Our biggest possibility is $n_{10} = 14$. Then $2019 - \binom{14}{10} = 2019 - \binom{14}{4} = 2019 - 1001 = 1018$. Continuing, we choose $n_9 = 13$, and we get $1018 - \binom{13}{9} = 303$. Then $n_8 = 11$ and $303 - \binom{11}{8} = 138$. Then $n_7 = 10$ and $138 - \binom{10}{7} = 18$. Then $n_6 = 7$ and $18 - \binom{7}{6} = 11$. Then $n_5 = 6$ and $11 - \binom{6}{5} = 5$. Then $n_4 = 5$ and $5 - \binom{5}{4} = 0$, so we are done. The desired quantity is $5 + 6 + 7 + 10 + 11 + 13 + 14 = 66$.

18. If n is a randomly chosen integer between 1 and 390 (inclusive), what is the probability that $26n$ has more positive factors than $6n$?

Answer: $\frac{123}{390}$

Solution: For an integer x, let $\psi(x)$ denote the number of positive factors of x. Write $n = 2^a * 3^b * 13^c * d$, where d is not divisible by 2, 3, or 13. Then $\psi(26n) = (a+1)(b)(c+1)\psi(d)$ and $\psi(6n) = (a+1)(b+1)(c)\psi(d)$, so we really want to know the probability that $(b)(c+1) > (b+1)(c)$, i.e., $b > c$. It's actually easier to count the integers n where $c \geq b$. For $c = 0$ and $b = 0$, there are $(390)(\frac{12}{13})(\frac{2}{3}) = 240$ possibilities. There are $(390)(\frac{1}{13}) = 30$ integers between 1 and 390 that are divisible by 13 (i.e., that have $c \geq 1$). Of these, the only ones that are divisible by 9 (i.e., that have $b \geq 2$) are $13 * 9 = 117$, $13 * 9 * 2 = 234$, and $13 * 9 * 3 = 351$. Since neither none of these is divisible by 13^2, these numbers have $b \geq 2 > c$, but the other 27 have $c \geq 1 \geq b$. So the answer is $\frac{390-(240+27)}{390} = \frac{123}{390}$.

. . . . Mathathon Round 7 (9 points each) . . . .

19. Suppose S is an n-element subset of $\{1, 2, 3, ..., 2019\}$. What is the largest possible value of n such that for every $2 \leq k \leq n$, k divides exactly $n - 1$ of the elements of S?

Answer: 7

Solution: The set $\{35, 60, 84, 420, 840, 1260, 1680\}$ satisfies the desired property for $n = 7$, and we claim that there is no solution set S for $n = 8$. Suppose (for the sake of contradiction) that there is such a set $S = \{s_1, \ldots, s_8\}$. Since 2 divides exactly 7 elements of S, there is one element (say, s_1) which is odd. Note that 2, 4, 6, and 8 must all divide s_2 through s_8 since none of them divides s_1. Since 3 divides 7 elements of S, it must divide at least 6 of s_2 through s_8, say, s_3 through s_8. By the same token, 5 divides s_4 through s_8 (without loss of generality) and 7 divides s_5 through s_8. Then s_5 through s_8 are all divisible by $(2, 3, 4, 5, 6, 7, 8) = 840$. But 840 and 1680 are the only multiples of 840 between 1 and 2019, so this is impossible.

20. For each positive integer n, let $f(n)$ be the fewest number of terms needed to write n as a sum of factorials. For example, $f(28) = 3$ because $4! + 2! + 2! = 28$ and 28 cannot be written as the sum of fewer than 3 factorials. Evaluate $f(1) + f(2) + \cdots + f(720)$.

Answer: 5401

Solution: Let $g(k) = f(0) + f(1) + \cdots + f(k! - 1)$. Note that $g(1) = f(0) = 0$. For $k \geq 2$, applying the (obviously optimal) greedy algorithm gives

$$g(k) = [f(0) + \cdots + f((k-1)! - 1)] + [f((k-1)!) + \cdots + f(2(k-1)! - 1)] + \cdots + [f((k-1)(k-1)!) + \cdots + f(k(k-1)! - 1)],$$

so

$$g(k) = [g(k-1)] + [(k-1)! + g(k-1)] + \cdots + [(k-1)(k-1)! + g(k-1)],$$

i.e.,

$$g(k) = kg(k-1) + \frac{k(k-1)}{2} \cdot (k-1)!.$$

We can now compute $g(2) = 1$, $g(3) = 9$, $g(4) = 72$, $g(5) = 600$, $g(6) = 5400$. (Alternatively, induct on the recurrence to get $g(k) = k! \cdot \frac{k(k-1)}{4}$.) Note that since $f(0) = 0$ and $f(720) = 1$, the desired quantity is $5400 + 1 = 5401$.

21. Evaluate $\sum_{n=1}^{\infty} \frac{\phi(n)}{101^n - 1}$, where $\phi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n.

Answer: $\frac{101}{10000}$

Solution: We have $\frac{1}{101^n - 1} = \frac{101^{-n}}{1 - 101^{-n}} = \sum_{k=1}^{\infty} 101^{-kn}$, so the desired sum is

$$\sum_{n=1}^{\infty} \phi(n) \sum_{k=1}^{\infty} 101^{-kn}.$$

Since everything in sight is positive, we can regroup the terms to get

$$\sum_{m=1}^{\infty} \sum_{n|m} \phi(n) 101^{-m}.$$

Using $\sum_{n|m} \phi(n) = m$, we see that this equals $\sum_{m=1}^{\infty} m 101^{-m}$, which is a standard sum and can be evaluated to get $\frac{101}{100^2} = \frac{101}{10000}$.