# Math Majors of America Tournament for High Schools
## 2016 Mixer Solutions

1. Give a fake proof that $0 = 1$ on the back of this page. The most convincing answer to this question at this test site will receive a point.

**Answer: subjective**

**Solution:** to be decided by graders

2. It is often said that once you assume something false, anything can be derived from it. You may assume for this question that $0 = 1$, but you can only use other statements if they are generally accepted as true or if your prove them from this assumption and other generally acceptable mathematical statements. With this in mind, on the back of this page prove that every number is the same number.

**Answer: See solution**

**Solution:** Sketch: For any $a$ and $b$, $a - b = a \cdot 1 - b \cdot 1 = a \cdot 0 - b \cdot 0 = 0 - 0 = 0$. Hence, every $a$ and $b$ are equal.

3. Suppose you write out all integers between 1 and 1000 inclusive. (The list would look something like $1, 2, 3, \ldots, 10, 11, \ldots, 999, 1000$.) Which digit occurs least frequently?

**Answer: 0**

**Solution:** All the other digits occur the same number of times besides 1, which occurs the most. 0 isn't used as a leading digit, so we use 0 least frequently.

4. Pick a real number between 0 and 1 inclusive. If your response is $r$ and the standard deviation of all responses at this site to this question is $\sigma$, you will receive $r(1 - (r - \sigma)^2)$ points.

**Answer: TBD**

**Solution:** To be calculated at the test site

5. Find the sum of all possible values of $x$ that satisfy $243^{x+1} = 81^{x^2+2x}$.

**Answer:** $-\frac{3}{4}$

**Solution:** Converting both expressions to powers of 3, we have $3^{5x+5} = 3^{4x^2+8x}$, which yields to $5x + 5 = 4x^2 + 8x$. Rearranging gives $4x^2 + 3x - 5 = 0$. The sum of the roots of a quadratic $ax^2 + bx + c = 0$ is $-\frac{b}{a}$, so the sum of the roots of $4x^2 + 3x - 5$ is $-\frac{3}{4}$.

6. How many times during the day are the hour and minute hands of a clock aligned?

**Answer: 22**

**Solution:** There are 22 such times – 11 every time the clock goes around (by the time the clock gets back to 12, it is in the next 12 hour period. The times are at about 12:00, 1:05, 2:11, 3:16, 4:22, 5:27, 6:33, 7:38, 8:44, 9:49, 10:55 both am and pm.

7. A group of $N + 1$ students are at a math competition. All of them are wearing a single hat on their head. $N$ of the hats are red; one is blue. Anyone wearing a red hat can steal the blue hat, but in the process that person's red hat disappears. In fact, someone can only steal the blue hat if they are wearing a red hat. After stealing it, they would wear the blue hat. Everyone prefers the blue hat over a red hat, but they would rather have a red hat than no hat at all. Assuming that everyone is perfectly rational, find the largest prime $N$ such that nobody will ever steal the blue hat.

**Answer: 2**

**Solution:** First consider the case when the number of red hats is 2. Person 1 has the blue hat, and people 2 and 3 have red hats. If 2 steals 1's hat, then 3 steals the blue hat and is safe. So neither 2 nor 3 will steal 1. Therefore, someone will steal the blue hat if there are 4 people, since then it would reduce to the previous case where the blue hat is safe. Whenever the number of red hats is even, nobody will steal the blue hat. When the number of red hats is odd, whoever gets to the blue hat first will steal it. We can see this by further case analysis and prove the result by induction.

8. On the back of this page, prove there is no function $f(x)$ for which there exists a (finite degree) polynomial $p(x)$ such that $f(x) = p(x)(x + 3) + 8$ and $f(3x) = 2f(x)$.

**Answer: See solution**

**Solution:** Suppose such an f exists. It is a polynomial. Say its degree is $n$ and its leading coefficient is $a$. The leading coefficient of $f(3x)$ is $(3^n)a$, and the leading coefficient of $2f(x)$ is $2a$. [Give the point if they compare leading coefficients, or for any other correct proof.]

9. Given a cyclic quadrilateral $YALE$ with $YA = 2, AL = 10, LE = 11, EY = 5$, what is the area of $YALE$?

**Answer: 36**

**Solution:** $YALE$ has the same area as the quadrilateral $Y'ALE$ where $Y'A = YE = 5, Y'E = YA = 2$. $Y'A^2 + AL^2 = 125 = LE^2 + EY'^2$. (Here $Y'$ is the point $Y$ reflected over the perpindicular bisector of $AE$.) So triangles $Y'AL$ and $LEY'$ are right-angled and their areas are 25 and 11. Hence the total area of $YALE$ is $36$.

10. About how many pencils are made in the U.S. every year? If your answer to this question is $p$, and our (good) estimate is $\rho$, then you will receive $\max(0, 1 - \frac{1}{2}|\log_{10}(p) - \log_{10}(\rho)|)$ points.

**Answer: 2 billion**

**Solution:** This is according to https://www.pencils.net/didyaknow.cfm. We could estimate this as follows: There are about 300 million people in the U.S. People on average buy a pack of pencils once every three years. So in any given years, there are 100 million packs purchased. Each pack has about 10 pencils in it. So we are purchasing about 1 billion pencils. (Of course this isn't completely accurate because many pencils are made outside of the U.S. (we only manufacture about a fifth of the pencils), but one could imagine more pencils in the U.S. being consumed. This is simply a rough estimate and would get us about .65 points for this question).

11. The largest prime factor of $520, 302, 325$ has 5 digits. What is this prime factor?

**Answer: 10613**

**Solution:** We quickly notice that $520302325$ is divisible by $5$, resulting in the quotient $104060465$. Using the Sophie-Germain Identity, we can rewrite this as $(100 + 1)^4 + 4 \cdot 2^4 = (a^2 2ab + 2b^2)(a^2 + 2ab + 2b^2) = 9805 \cdot 10613$. Thus, the 5-digit prime factor is $10613$.

12. The previous question was on the individual round from last year. It was one of the least frequently correctly answered questions. The first step to solving the problem and spotting the pattern is to divide $520, 302, 325$ by an appropriate integer. Unfortunately, when solving the problem many people divide it by $n$ instead, and then they fail to see the pattern. What is $n$?

**Answer: 25**

**Solution:** If we first divide by 25, we fail to see the Sophie-Germain factorization. Most of the test editors that had difficulty solving this question last year admitted that this was the hurdle that they faced.