. . . . Mathathon Round 1 (2 points each) . . . .

1. This year, the Mathathon consists of 7 rounds, each with 3 problems. Another math test, Aspartaime, consists of 3 rounds, each with 5 problems. How many more problems are on the Mathathon than on Aspartaime?

Answer: 6

Solution: Mathathon has $7 \cdot 3 = 21$ problems, and Aspartaime has $3 \cdot 5 = 15$ problems, so Mathathon has $21 - 15 = 6$ more problems than Aspartaime.

2. Let the solutions to $x^3 + 7x^2 - 242x - 2016 = 0$ be a, b, and c. Find $a^2 + b^2 + c^2$. (You might find it helpful to know that the roots are all rational.)

Answer: 533

Solution: Solution 1: $(a+b+c)^2 - 2(ab+bc+ca) = a^2+b^2+c^2$. By Vieta's Formulas, $a+b+c = -7$, and $ab+bc+ca = -242$, so the sum of the squares of the roots is $49 - 2(-242) = 49 + 484 = 533$.
Solution 2: $x^3 + 7x^2 - 242x - 2016 = (x+9)(x+14)(x-16)$, so the sum of the squares of the roots is $81 + 196 + 256 = 533$.

3. For triangle ABC, you are given $AB = 8$ and $\angle A = 30°$. You are told that BC will be chosen from amongst the integers from 1 to 10, inclusive, each with equal probability. What is the probability that once the side length BC is chosen there is exactly one possible triangle ABC?

Answer: $\frac{2}{5}$

Solution: Let $x = BC$. By the Law of Sines, we have $\frac{\sin 30°}{x} = \frac{\sin C}{8}$. Notice for x ¡ 4, we have $\sin C = \frac{4}{x} > 1$, which is impossible. For $x = 4$, we have $\sin C = 1$, which uniquely determines ABC. For $x = 5, 6$, or 7, we have $\sin C = \frac{4}{x} > \frac{1}{2}$, so C is either greater than 30° or less than 150°, both of which are possible, so $x = 5, 6$, or 7 do not uniquely determine ABC. For $x = 8, 9$, or 10, $\sin C = \frac{4}{x} < \frac{1}{2}$, which means C is either no more than 30° or at least 150°. Since C cannot be at least 150, $x = 8, 9$, or 10 uniquely determines ABC. So out of the 10 possibilities, $4, 8, 9$, and 10 all work. So the probability is $\frac{4}{10} = \frac{2}{5}$.

. . . . Mathathon Round 2 (3 points each) . . . .

4. It's raining! You want to keep your cat warm and dry, so you want to put socks, rain boots, and plastic bags on your cat's four paws. Note that for each paw, you must put the sock on before the boot, and the boot before the plastic bag. Also, the items on one paw do not affect the items you can put on another paw. How many different orders are there for you to put all twelve items of rain footwear on your cat?

Answer: 369600

Solution: Imagine the items for each paw are color-coded (blue sock, boot, and bag go on the left front paw, for example). For each item, you must choose what color item you will put on your cat next. There are a total of 4 colors, and 3 items of each color, so the question is equivalent to the number of distinct ways to arrange 3 B, 3 R, 3 Y, and 3 G in a line. There are $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = 220 \cdot 84 \cdot 20 \cdot 1 = 369600$ orders.

5. Let a be the square root of the least positive multiple of 2016 that is a square. Let b be the cube root of the least positive multiple of 2016 that is a cube. What is $a - b$?

Answer: 84

Solution: 2016 is $2^5 \cdot 3^2 \cdot 7$. The least multiple of 2016 that is a square is $2^6 \cdot 3^2 \cdot 7^2$, so $a = 2^3 \cdot 3 \cdot 7 = 8 \cdot 21 = 168$. The least multiple of 2016 that is a cube is $2^6 \cdot 3^3 \cdot 7^3$, so $b = 2^2 \cdot 3 \cdot 7 = 4 \cdot 21 = 84$. The difference of a and b is 84.

6. Hypersomnia Cookies sells cookies in boxes of 6, 9 or 10. You can only buy cookies in whole boxes. What is the largest number of cookies you cannot exactly buy? (For example, you couldn't buy 8 cookies.)

Answer: 23

Solution: It can be checked that 23 is not attainable. $24 = 4 \cdot 6$, $25 = 6 + 9 + 10$, $26 = 2 \cdot 10 + 6$, $27 = 3 \cdot 9$, $28 = 10 + 3 \cdot 6$, $29 = 2 \cdot 10 + 9$ and every larger number can be achieved by adding a multiple of 6.

. . . . Mathathon Round 3 (4 points each) . . . .

7. There is a store that sells each of the 26 letters. All letters of the same type cost the same amount (i.e. any 'a' costs the same as any other 'a'), but different letters may or may not cost different amounts. For example, the cost of spelling "trade" is the same as the cost of spelling "tread," even though the cost of using a 't' may be different from the cost of an 'r.' If the letters to spell out 1 cost \$1001, the letters to spell out 2 cost \$1010, and the letters to spell out 11 cost \$2015, how much do the letters to spell out 12 cost?

Answer: 2024

Solution: The letters in the words one and twelve are exactly the same as the letters in the words two and eleven,, so the cost of 2 and 11 equals the cost of 1 and 12, so the cost of 12 is $1010 + 2015 - 1001 = 2024$.

8. There is a square $ABCD$ with a point P inside. Given that $PA = 6$, $PB = 9$, $PC = 8$. Calculate PD.

Answer: $\sqrt{19}$

Solution: Let the length of square be t. Draw a perpendicular line from P to AB, BC, CD, DA so that the point of intersection is $EFGH$. So that $AE = x, EB = t - x, BF = y, FC = t - y, CG = t - x, GD = x, DH = t - y, HA = y$. It is clear that $a^2 = x^2 + y^2, b^2 = (t - x)^2 + y^2, c^2 = (t - x)^2 + (t - y)^2, d^2 = (t - y)^2 + x^2$. From the four equations, it is clear that $a^2 + c^2 = b^2 + d^2$. Hence $d = \sqrt{(19)}$. (Note that $a = PA, b = PB, c = PC, d = PD$.)

9. How many ordered pairs of positive integers (x, y) are solutions to $x^2 - y^2 = 2016$?

Answer: 12

Solution: $2016 = (x + y)(x - y)$, and $2016 = 2^5 \cdot 3^2 \cdot 7$. Since $x + y$ and $x - y$ must have the same parity (both must be even or both must be odd), and 2016 is even, both $x + y$ and $x - y$ must be even. Now, let us count the number of ways to distribute the factors 2^3, 3^2, and 7 between $x + y$ and $x - y$. There are 4 ways to distribute the two's ($x + y$ can have $0, 1, 2,$ or 3 powers of 2, and then $x - y$ gets the remaining factors), 3 ways to distribute the three's, and 2 ways to distribute the seven, which yields 24 pairs. However, we must have that $x + y > x - y$ (or one of x or y is negative). Half of the pairs $(x + y, x - y)$ have $x + y > x - y$, so we have a total of 12 suitable pairs.

. . . . Mathathon Round 4 (6 points each) . . . .

10. Given a triangle with side lengths 5, 6 and 7, calculate the sum of the three heights of the triangle.

Answer: $\frac{214\sqrt{6}}{35}$

Solution: Using Heron's formula, area= $\sqrt{s(s - a)(s - b)(s - c)}$ where s is half the perimeter. In this case, the area of the triangle is $6\sqrt{6}$. Hence the three heights are $12\frac{\sqrt{6}}{5}, 12\frac{\sqrt{6}}{6}, 12\frac{\sqrt{6}}{7}$ and the sum is hence $214\frac{\sqrt{6}}{35}$. We could also have used the law of cosines and an alternate method for finding the area.

11. There are 6 people in a room. Each person simultaneously points at a random person in the room that is not him/herself. What is the probability that each person is pointing at someone who is pointing back to them?

Answer: $\frac{3}{3125}$

Solution: Note that the condition is the same as saying that everyone is "paired up" that is, there are 3 pairs of students, both of the students in each pair pointing to each other.

Solution 1: The first person to point has a probability 1 of pointing to a valid person. That person has a $\frac{1}{5}$ chance of pointing to the first person. The 3rd person has a $\frac{3}{5}$ probability of pointing to someone not in the first pair, and the person he/she picks has a $\frac{1}{5}$ probability of pointing back at him/her. Then the last 2 people each have a $\frac{1}{5}$ chance of pointing to each other, for a total probability of $1 \cdot \frac{1}{5} \cdot \frac{3}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} = \frac{3}{3125}$.

Solution 2: Count the total number of possible pair configurations. It is $\binom{6}{2}\binom{4}{2}\binom{2}{2}/3!$, which comes from $\binom{6}{2}$ for the first pairing, $\binom{4}{2}$ for the second, $\binom{2}{2}$ for the third, divided by 3! for the order. This is just $15 \cdot 6 \cdot \frac{1}{6} = 15$. The total number of pointing possibilities is 5^6, so the answer is $15/5^6 = 3/5^5 = 3/3125$.

12. Find all x such that $\sum_{i=0}^{\infty} ix^i = \frac{3}{4}$

Answer: $\frac{1}{3}$

Solution: Note that $\sum_{i=0}^{\infty} ix^i = x(1 + 2x + 3x^2 + \cdots) = x(\frac{1}{1-x} + x(1 + 2x + 3x^2 + \cdots))$. Repeating this process, we find that $\sum_{i=0}^{\infty} ix^i = x(\sum_{i=0}^{\infty} \frac{x^i}{1-x}) = \frac{x}{(1-x)^2} = \frac{3}{4}$, so $(3x-1)(x-3) = 0$. We note that $x \neq 3$ as then the sequence would not converge, so $x = \frac{1}{3}$.

.... Mathathon Round 5 (7 points each) ....

13. Let $\{a\}_{n\geq 1}$ be an arithmetic sequence, with $a_1 = 0$, such that for some positive integers k and x we have $a_{k+1} = \binom{k}{x}$, $a_{2k+1} = \binom{k}{x+1}$, and $a_{3k+1} = \binom{k}{x+2}$. Let $\{b\}_{n\geq 1}$ be an arithmetic sequence of integers with $b_1 = 0$. Given that there is some integer m such that $b_m = \binom{k}{x}$, what is the number of possible values of b_2?

Answer: 8

Solution: We call $\binom{n}{x}$ = w. Further, we know that the difference between each of the given numbers is the same since a is an arithmetic sequence. Thus, $\binom{n}{x+1} = 2w$ and $\binom{n}{x+2} = 3w$. From this, we can conclude that $\frac{\binom{n}{x+1}}{\binom{n}{x}} = 2$ and that $\frac{\binom{n}{x+2}}{\binom{n}{x}} = 3$. We have a system of two variables and two equations, which we can solve. Solving, we find n to be 14 and x to be 4. Thus, $a_{n+1} = \binom{14}{4} = 1001$. As a result, $b_2 - b_1 = b_2$ can be any of the factors of 1001, of which there are $\boxed{8}$ since $1001 = 7 \cdot 11 \cdot 13$ and each of these prime factors either appears or does not in one of the factors of 1001.

14. Let $A = \arcsin(\frac{1}{4})$ and $B = \arcsin(\frac{1}{7})$. Find $\sin(A+B)\sin(A-B)$.

Answer: $\frac{33}{784}$

Solution: Note that $\sin(A+B)\sin(A-B) = (\sin A)^2 - (\sin B)^2$, so $\sin(A+B)\sin(A-B) = \frac{1}{16} - \frac{1}{49} = \frac{33}{784}$.

15. Let $\{f_i\}_{i=1}^9$ be a sequence of continuous functions such that $f_i : \mathbb{R} \to \mathbb{Z}$ is continuous (i.e. each f_i maps from the real numbers to the integers). Also, for all i, $f_i(i) = 3^i$. Compute $\sum_{k=1}^{9} f_k \circ f_{k-1} \circ \cdots \circ f_1(3^{-k})$.

Answer: 29523

Solution: In order for f_i to be continuous, f_i must be constant. So $f_i \equiv 3^i$. Then the compositions in the summation are irrelevant, and all that matters is the final application of f_k. So the sum is $\sum_{k=1}^{9} 3^k = \frac{3}{2}(3^9 - 1) = 29523$.

.... Mathathon Round 6 (8 points each) ....

16. If x and y are integers for which $\frac{10x^3 + 10x^2y + xy^3 + y^4}{203} = 1134341$ and $x - y = 1$, then compute $x + y$.

Answer: 203

Solution: Notice that $1134341 = 1030301 + 10(10401) = 101^3 + 10(102^2)$.
Factoring the polynomial in the numerator of the fraction on the LHS gives $(x+y)(10x^2 + y^3)$.
Finally we notice that $101 + 102 = 203$. This gives us our answer of $x = 102$ and $y = 101$.

17. Let T_n be the number of ways that n letters from the set $\{a, b, c, d\}$ can be arranged in a line (some letters may be repeated, and not every letter must be used) so that the letter a occurs an odd number of times. Compute the sum $T_5 + T_6$.

Answer: 2512

Solution: A linear arrangement of n letters from the set $A = \{a, b, c, d\}$ is called a word of length n over A. Let B_n be the set of words of length n over A in which a occurs an odd number of times (so $T_n = |B_n|$). Let $\tilde{B}_n$ be the set of words of length n over A in which a occurs an even number of times (so $|\tilde{B}_n| = 4^n - T_n$). To construct an element of B_{n+1}, we can either append a b, c, or d to the end of an element of B_n or append an a to the end of an element of $\tilde{B}_n$. This construction produces every element of B_{n+1} exactly once, so

$$T_{n+1} = 3T_n + (4^n - T_n) = 4^n + 2T_n.$$

Using the fact that $T_1 = 1$, it follows easily by induction that $T_n = 2^{n-1}(2^n - 1)$ for all positive integers n. In particular, $(T_5, T_6) = (2^{5-1}(2^5 - 1), 2^{6-1}(2^6 - 1)) = (496, 2016)$. So the answer is 2512.

18. McDonald plays a game with a standard deck of 52 cards and a collection of chips numbered 1 to 52. He picks 1 card from a fully shuffled deck and 1 chip from a bucket, and his score is the product of the numbers on card and on the chip. In order to win, McDonald must obtain a score that is a positive multiple of 6. If he wins, the game ends; if he loses, he eats a burger, replaces the card and chip, shuffles the deck, mixes the chips, and replays his turn. The probability that he wins on his third turn can be written in the form $\frac{x^2 \cdot y}{z^3}$ such that x, y, and z are relatively prime positive integers. What is $x + y + z$? (NOTE: Use Ace as 1, Jack as 11, Queen as 12, and King as 13)

Answer: 338

Solution: Let m_n represent a multiple of n; $P(m_x|m_y)$ is defined as the probability of card being multiple of x and chip being multiple of y for some $(x, y) \in N \setminus 0$. $P(win) = P((m_2|m_3) \setminus m_6) + P((m_3|m_2) \setminus m_6) + P(m_6|any) + P(any|m_6) - P(m_6|m_6) = \frac{4 \times (4 \times 9 + 2 \times 18 + 2 \times 52 + 8 \times 13 - 8 \times 2)}{52^2} = \frac{66}{169}$ So the probability that McDonald wins on his third turn is $\frac{103}{169} \times \frac{103}{169} \times \frac{66}{169} = \frac{(103)^2(66)}{169^3}$ and so $x + y + z = 103 + 169 + 66 = 338$.

.................................Mathathon Round 7 (9 points each).................................

19. Let $f_n(x) = \ln(1 + x^{2^n} + x^{2^{n+1}} + x^{3 \cdot 2^n})$. Compute $\sum_{k=0}^{\infty} f_{2k}\left(\frac{1}{2}\right)$.

Answer: $\ln(2)$

Solution: Note $\ln(1 + x^{2^n} + x^{2^{n+1}} + x^{3 \cdot x^{2^n}}) = \ln(1 + x^{2^n}) + \ln(1 + x^{2^{n+1}})$. Observe that $\sum_{k=0}^{\infty} f_{2k}(\frac{1}{2}) = \ln(\prod_{k=0}^{\infty} 1 + (\frac{1}{2})^{2^k})$. Consider $\prod_{k=0}^{\infty} 1 + x^{2^k}$. As any positive may be uniquely represented in binary, x^l will be in the expansion of this product exactly once for every integer l, $\prod_{k=0}^{\infty} 1 + x^{2^k} = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, so $\ln(\prod_{k=0}^{\infty} 1 + (\frac{1}{2})^{2^k}) = \ln(\frac{1}{1 - \frac{1}{2}}) = \ln(2)$.

20. $ABCD$ is a quadrilateral with $AB = 183$, $BC = 300$, $CD = 55$, $DA = 244$, and $BD = 305$. Find AC.

Answer: 273

Solution: ABD is a right triangle $(3 - 4 - 5)$ and CBD is a right triangle $(11 - 60 - 61)$. Therefore, $\angle BAD$ and $\angle BCD$ are right angles, so $ABCD$ is a cyclic quadrilateral. By Ptolemy's Theorem, $AC \cdot BD = AB \cdot CD + BC \cdot AD$, so plugging in values yields $AC = 273$.

21. Define $\overline{xyz(t+1)} = 1000x + 100y + 10z + t + 1$ as the decimal representation of a four digit integer. You are given that $3^x 5^y 7^z 2^t = \overline{xyz(t+1)}$ where x, y, z, and t are non-negative integers such that t is odd and $0 \le x, y, z, (t+1) \le 9$. Compute $3^x 5^y 7^z$.

Answer: 63

Solution: First, suppose that $y \geq 1$. This is impossible, since this leads only to $t = 4$ because the whole thing must be a multiple of 5. Thus, $y = 0$. By the bound $\overline{xyz(t+1)} \leq 10,000$, we obtain $t \leq 12, z \leq 4, x \leq 8$. Notice also that since we have restricted ourselves to even numbers and digits, we have $t \in \{1, 3, 5, 7\}$.

Case 1: $z = 4$. Since $7^4 = 2401$, then $x \geq 2$, but $2401 \cdot 9 > 10,000$, which exceeds the bound – a contradiction.

Case 2: $z = 3$.

- $t = 1$. Then $3^x \times 343 \times 2 = \overline{x032}$. Clearly, $x \geq 1$. Analyzing modulo 3 (recall that a number is equivalent to the sum of its digits modulo 3), there are only 3 possible options for x, namely $x \in \{1, 4, 7\}$. The last two give numbers greater than 10,000. The case $x = 1$ does not lead to a solution.
- $t > 1$. Then the left hand side is divisible by 4. Looking at the last two digits, $\overline{3(t+1)}$ must be divisible by 4, i.e. $t = 1$ or $t = 5$. The first case has already been examined. The latter does not lead to any solution.

Case 3: $z = 2$

- $t = 1$. Then $3^x \cdot 49 \cdot 2 = \overline{x022}$ Modulo 3, we have $x \in \{2, 5, 8\}$. The latter two yield numbers exceeding 10,000. The case $x = 2$ gives a number bellow 1,000.
- $t > 1$. Then the left hand side is divisible by 4. Looking at the last two digits, $\overline{2(t+1)}$ must be divisible by 4, i.e. $t = 3$ or $t = 7$. Neither of them lead to a solution.

Case 4: $z = 1$.

- $t = 1$. Then $3^x \cdot 7 \cdot 2 = \overline{x012}$. Clearly, $x \geq 1$. Analyzing modulo 3 again, we have $x \in \{3, 6\}$. The first produces a number below 1000, the second above 10,000.
- $t > 1$. Then the left hand side is divisible by 4. Looking at the last two digits, $\overline{1(t+1)}$ must be divisible by 4, i.e. $t = 1$ or $t = 5$. The first one has already been examined. If $t = 5$. then we obtain the solution $3^2 5^0 7^1 2^5 = 2016$ (which is nice!).

Case 5: $z = 0$. Again, checking separately the cases $t = 1$ and $t > 1$, and arguing as above by considering modulo 3 and 4, this case does not lead to a new solution.