

Math Majors of America Tournament for High Schools
2016 Individual Solutions



1. For what value of x is the function $f(x) = (x - 2)^2$ minimized?

Answer: 2

Solution: The function has a double root at $x = 2$, and it is positive elsewhere. So the minimum occurs when $x = 2$.

2. Two spheres A and B have centers $(0, 0, 0)$ and $(2016, 2016, 1008)$. Sphere A has a radius of 2017. If A and B are externally tangent, what is the radius of sphere B ?

Answer: 1007

Solution: The distance between the centers A and B can be calculated with the Pythagorean theorem and is $3 \cdot 1008 = 3024$. Therefore the radius of B must be $3024 - 2017 = 1007$.

3. Consider a white, solid cube of side length 5 made of $5 \times 5 \times 5 = 125$ identical unit cubes with faces parallel to the faces of the larger cube. The cube is submerged in blue paint until the entire exterior of the cube is painted blue, so that a face of a smaller cube is blue if and only if it is part of a face of the larger cube. A random smaller cube is selected and rolled. What is the probability that the up-facing side is blue?

Answer: $\frac{1}{5}$

Solution: There are 8 cubes with 3 sides painted blue, 36 cubes with 2 sides painted blue, 54 cubes with 1 side painted blue, and 27 cubes with no sides painted blue. The probability is $\frac{8}{125} \cdot \frac{1}{2} + \frac{36}{125} \cdot \frac{1}{3} + \frac{54}{125} \cdot \frac{1}{6} = \frac{150}{750} = \frac{1}{5}$.

4. [This problem was thrown out.]

Answer: N/A

Solution: This problem was thrown out on the day of the contest during grading. Nobody received points for this question.

5. Find the largest prime factor of 4003997, given that 4003997 is the product of two primes.

Answer: 2003

Solution: $4003997 = 2001^2 - 2^2 = (2001 - 2)(2001 + 2) = 1999 \cdot 2003$. 2003 is prime, so the largest prime factor of 4003997 is 2003.

6. Elaine is writing letters to six friends. She has six addressed letters and six addressed envelopes. She puts each letter randomly into an envelope without first checking the name on the envelope. What is the probability that exactly one envelope has the correct letter?

Answer: $\frac{11}{30}$

Solution: The number of ways to put letters into envelopes is $6! = 720$. The number of ways of putting exactly one letter in the right envelope is $6 \cdot D_5$, where D_5 is the number of arrangements where 5 letters are not in the right envelopes (This is called a derangement). We can compute D_5 recursively. Pick an envelope $i \neq 1$ arbitrarily. We either put the first letter in i , or we do not. If we do, then this reduces the problem to finding D_3 , since the intended letter for i cannot go into that envelope. If we do not, then this reduces the problem to finding D_4 , since we now have 4 letters and 4 envelopes, and each letter cannot be matched to exactly one of the envelopes (we are assuming that letter i cannot go in envelope 1). There are 4 choices for i , so we see that $D_5 = 4 \cdot (D_3 + D_4)$. Continuing recursively in this fashion, we get that the number of derangements for 5 items is 44. Therefore, the probability of having exactly 1 letter in the correct envelope is $6 \cdot \frac{44}{720} = \frac{11}{30}$.

7. Colin has written the numbers $1, 2, \dots, n$ on a chalk board. He will erase at most 4 of the numbers (he might choose not to erase any of the numbers) and then circle $n - 4$ of the remaining numbers. There are exactly 2016 possible ways to do this. Find n . (You should assume that circling the same set of numbers but erasing different numbers should count as different possible ways.)

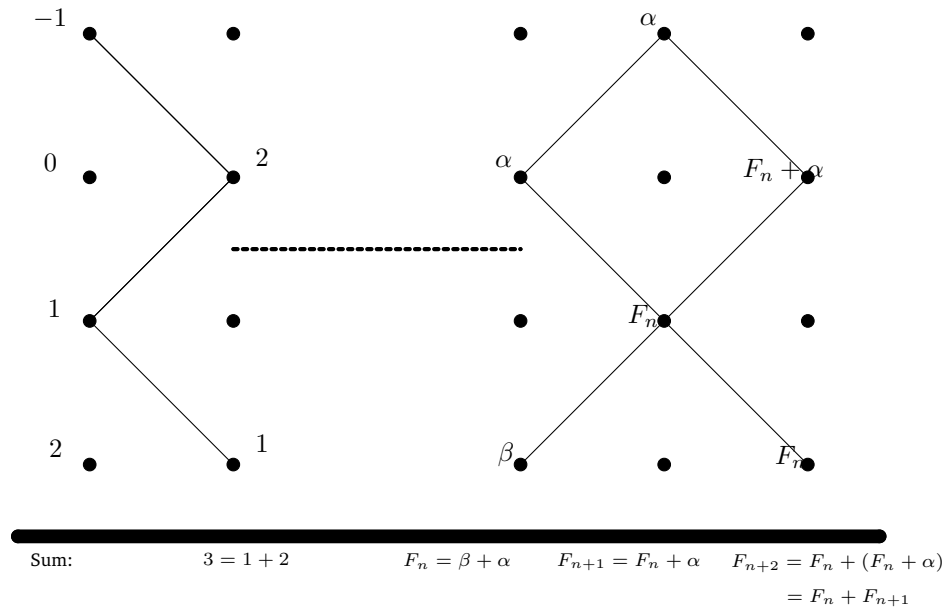
Answer: 9

Solution: Colin might as well circle the numbers that he wants to circle and then erase at most 4 of the uncircled numbers. The number of ways to circle $n - 4$ numbers is $\binom{n}{n-4} = \binom{n}{4}$. Each of the 4 uncircled numbers can then either be erased or left on the board. Therefore, $2016 = \binom{n}{4} \cdot 2^4$. This implies that $\binom{n}{4} = 126$, so $n = 9$.

8. Let $R_n = r_1, r_2, r_3, \dots, r_n$ be a finite sequence of integers such that for all possible i , r_i is either $-1, 0, 1$ or 2 . Furthermore, for all i such that $1 \leq i < n$, r_i and r_{i+1} have opposite parity (i.e. one is odd and the other is even). Finally, -1 and 2 do not occur adjacently in the sequence. Given that r_1 must be even (i.e. either $r_1 = 0$ or $r_1 = 2$), $S(n)$ is the number of possible sequences R_n could be. For example, $S(1) = 2$. For what k is $S(k) = 12^2$?

Answer: 10

Solution:



In the diagram, $S(n)$ is written as $F(n)$. The diagram also starts from -1 and 1 , but the solution is symmetric. We can start with 0 or 2 . If $r_1 = 2$, then we could have $2, 1$. If $r_1 = 0$, we could have $0, -1$ or $0, 1$. So from the first position we have one choice and from the other we have two. The number $S(n)$ will be the sum of the two different ways to get to that vertex. Since the “only one choice” option will be half of the “two choice option” the recurrence relation works out to be the Fibonacci relation as shown in the diagram. Since the initial conditions are shifted, we just need to count $S(10) = 144$.

9. What is the largest prime p for which the numbers $p^2 - 8$, $p^2 - 2$, and $p^2 + 10$ are all prime as well?

Answer: 7

Solution: Note that for $p = 2, 3, 5$, the statement does not hold. For $p = 7, 49 - 8, 49 - 2$, and $49 + 10$ are all prime. Now, for any number n , where n is not a multiple of 7 , n^2 is equivalent to $1, 2$, or 4 modulo 7 . If n^2 is $1 \pmod{7}$, then $n^2 - 8$ is divisible by 7 . If n^2 is $2 \pmod{7}$, then $n^2 - 8$ is divisible by 7 . If n^2 is $4 \pmod{7}$, then $n^2 + 10$ is divisible by 7 . Therefore, the largest such prime is 7 .

10. Express $\sqrt{25 + 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25}$ as an integer.

Answer: 2525

Solution: Divide the expression by 5 and square it to get $x(x+1)(x+2)(x+3) + 1$ with $x = 21$. This becomes $x^4 + 4x^3 + 6x^2 + 4x + 1$. Use the fact that this is a symmetric polynomial (or through other methods) to factor as $(x^2 + 3x + 1)^2$. This means our answer should

be $5(x^2 + 3x + 1)$. Plug in $x = 21$ to get 2525.

11. Let $ABCD$ be a quadrilateral where AC bisects $\angle A$, $AB \neq AD$, $BC = CD = 7$, and $AC \cdot BD = 36$. Find $AB + AD$.

Answer: $\frac{36}{7}$

Solution: Since $\angle BAC = \angle DAC$ and $BC = DC$, $\frac{\sin(\angle BAC)}{BC} = \frac{\sin(\angle DAC)}{DC}$. By the Law of Sines, $\frac{\sin(\angle ABC)}{AC} = \frac{\sin(\angle BAC)}{BC} = \frac{\sin(\angle DAC)}{DC} = \frac{\sin(\angle ADC)}{AC}$, so $\sin(\angle ABC) = \sin(\angle DAC)$. Since AB and AD are not equal, $\angle ABC$ is not equal to $\angle DAC$, but since they have the same sine, $\angle ABC = 180 - \angle DAC$, so the opposite angles of $ABCD$ add to 180, so $ABCD$ is cyclic. Therefore, by Ptolemy's Theorem, $AB \cdot CD + AD \cdot BC = AC \cdot BD$, so $7(AB + AD) = 36$, so $AB + AD = \frac{36}{7}$.

12. Find the largest integer x for which there is an integer y such that $x^4 + 12x^3 + 39x^2 + 17x - 57 = y^3$.

Answer: -3

Solution: Write the left hand side as $((x + 2)^3 + 3x^2 - 27)(x + 3) = y^3$. Clearly, both factors on the left are relatively prime. So both of them have to be perfect cubes. However, for $|x| > 3$, we clearly have $(x + 3)^3 > (x + 2)^3 + 3x^2 - 27 > (x + 2)^3$, so it is impossible to have a solution. The remaining cases when $|x| \leq 3$ are exhaustively checked.