Math Majors of America Tournament for High Schools

2015 Tiebreaker Test

Name: __________

ID : __________

Team : __________

– 75 minutes
– no calculators
– show reasoning

1. Each lattice point of the plane is labeled by a positive integer. Each of these numbers is the arithmetic mean of its four neighbors (above, below, left, right). Show that all the numbers are equal.

2. Determine, with proof, whether $22!6! + 1$ is prime.

3. Is there a number s in the set $\{\pi, 2\pi, 3\pi, \ldots, \}$ such that the first three digits after the decimal point of s are .001? Fully justify your answer.

4. For any nonnegative integer r, let S_r be a function whose domain is the natural numbers that satisfies

$$S_r(p^\alpha) = \begin{cases} 0, & \text{if } p \le r; \\ p^{\alpha-1}(p-r), & \text{if } p > r \end{cases}$$

for all primes p and positive integers α, and that $S_r(ab) = S_r(a)S_r(b)$ whenever a and b are relatively prime..

Now, suppose there are n squirrels at a party. Each squirrel is labeled with a unique number from the set $\{1, 2, \ldots, n\}$. Two squirrels are friends with each other if and only if the difference between their labels is relatively prime to n. For example, if $n = 10$, then the squirrels with labels 3 and 10 are friends with each other because $10 - 3 = 7$, and 7 is relatively prime to 10.

Fix a positive integer m. Define a *clique* of size m to be any set of m squirrels at the party with the property that any two squirrels in the clique are friends with each other. Determine, with proof, a formula (using S_r) for the number of cliques of size m at the squirrel party.