Math Majors of America Tournament for High Schools

2014 Mixer Solutions

1. How many real roots does the equation $2x^7 + x^5 + 4x^3 + x + 2 = 0$ have?

Answer: 1

Solution: Any polynomial of odd degree has at least one real root. By Descarte's rule of signs, there are no positive roots and at most one negative root.

2. Given that $f(n) = 1 + \sum_{j=1}^{n}(1 + \sum_{i=1}^{j}(2i+1))$, find the value of $f(99) - \sum_{i=1}^{99} i^2$.

Answer: $10,000$

Solution: $f(n)$ gives the sum of the first $n+1$ squares. The difference in the problem gives the value of $(n+1)^2$ which is thus $(100)^2 = 10,000$.

3. A rectangular prism with dimensions $1 \times a \times b$, where $1 < a < b < 2$, is bisected by a plane bisecting the longest edges of the prism. One of the smaller prisms is bisected in the same way. If all three resulting prisms are similar to each other and to the original box, compute ab. Note: Two rectangular prisms of dimensions $p \times q \times r$ and $x \times y \times z$ are similar if $\frac{p}{x} = \frac{q}{y} = \frac{r}{z}$.

Answer: 2

Solution: From the similarity relations, we have $1 : a : b = \frac{b}{2} : 1 : a = \frac{a}{2} : \frac{b}{2} : 1$, since $a < b < 2$. From these equations, we have $2a = b^2$ and $b = a^2$. Mutliplying these two equations together, we have $2ab = (a^2)(b^2)$. Dividing both sides by ab (which is allowed since $a, b > 0$), we get $2 = ab$. Thus, our answer is 2.

4. For fixed real values of p, q, r and s, the polynomial $x^4 + px^3 + qx^2 + rx + s$ has four non real roots. The sum of two of these roots is $4 + 7i$, and the product of the other two roots is $3 - 4i$. Compute q.

Answer: 71

Solution: Let m and n be the first two given roots. Since the coefficients are real, the non-real roots must be in complex conjugate pairs. Since $m + n$ is not real, m and n are not conjugates, thus we have m' and n' as the two conjugates, respectively. Hence $m \cdot n = 3 - 4i, m' + n' = 4 + 7i, m' \cdot n' = 3 + 4i$, and $m + n = 4 - 7i$. Using Vieta's formulas, we have that q is the second symmetric sum of hte polynomial (each of the roots multiplied two at a time). This factors to $(m + n)(m' + n') + mn + m'n' = 71$.

5. There are 10 seats in a row in a theater. Say we have an infinite supply of indistinguishable good kids and bad kids. How many ways can we seat 10 kids such that no two bad kids are allowed to sit next to each other?

Answer: 144

Solution: Let x_n be the number of ways to seat good and bad kids into a row of n chairs. Consider whether a good or bad kid sits in the leftmost chair. If it is a good kid, then anyone can sit in the adjacent spot, so there are x_{n-1} ways to fill the remaining seats. If it is a bad kid, then the chair adjacent must be seated with a good kid, and there are x_{n-2} ways to fill the remaining seats. Therefore, $x_n = x_{n-1} + x_{n-2}$ and as $x_1 = 2$, $x_n = F_{n+2}$. Then, $x_{10} = F_{12} = 144$.

6. There are an infinite number of people playing a game. They each pick a different positive integer k, and they each win the amount they chose with probability $\frac{1}{k^3}$. What is the expected amount that all of the people win in total?

Answer: $\frac{\pi^2}{6}$

Solution: Expected value of sum is sum of expected values. Each expected value is $\frac{1}{k^2}$. The answer is the sum of the squares of the reciprocals, which is known to be $\frac{\pi^2}{6}$.

7. There are 100 donuts to be split among 4 teams. Your team gets to propose a solution about how the donuts are divided amongst the teams. (Donuts may not be split.) After seeing the proposition, every team either votes in favor or against the propisition. The proposition is adopted with a majority vote or a tie. If the proposition is rejected, your team is eliminated and will never receive any

donuts. Another remaining team is chosen at random to make a proposition, and the process is repeated until a proposition is adopted, or only one team is left. No promises or deals need to be kept among teams besides official propositions and votes. Given that all teams play optimally to maximize the expected value of the number of donuts they receive, are completely indifferent as to the success of the other teams, but they would rather not eliminate a team than eliminate one (if the number of donuts they receive is the same either way), then how much should your team propose to keep?

Answer: 66

Solution: This is a variant of the Pirate's problem. We call the team making the proposition "the proposers." Let the teams, in reverse order of being hypothetically eliminated, be A, B, C, and D. Suppose there are only 2 teams left. Then the proposers can keep all the donuts in their own possession, and the tie vote will allow them to keep it all and give nothing to the other. When there are 3 teams left, the expected value of the two non-proposers (A, B) is 50 if they eliminate the proposers. So the proposer would have to offer one of them (A or B) 51 donuts to win the vote, and that team (C) will try to keep 49 for itself. In order for D to protect itself, it must consider the expectation of each of the other three teams. There is a $\frac{1}{3}$ chance at 49, $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$ chance at 51 (must not be the next proposed, and must be chosen by C to be compensated the 51), so $\frac{49}{3} + 17 \approx 34$ is the expected return of each team. So D should compensate at least one team by 34 to win their vote, and can keep 66 donuts for itself.

8. Dominic, Mitchell, and Sitharthan are having an argument. Each of them is either credible or not credible – if they are credible then they are telling the truth. Otherwise, it is not known whether they are telling the truth. At least one of Dominic, Mitchell, and Sitharthan is credible. Tim knows whether Dominic is credible, and Ethan knows whether Sitharthan is credible. The following conversation occurs, and Tim and Ethan overhear:

> Dominic: "Sitharthan is not credible."
> Mitchell: "Dominic is not credible."
> Sitharthan: "At least one of Dominic or Mitchell is credible."

Then, at the same time, Tim and Ethan both simultaneously exclaim: "I can't tell exactly who is credible!" They each then think for a moment, and they realize that they can. If Tim and Ethan always tell the truth, then write on your answer sheet exactly which of the other three are credible.

Answer: Mitchell

Solution: We consider the truth values (or credibility values; we just don't have that the opposite of false is true). The only possibilities that don't lead to a contradiction are T, F, F; F, T, T; and F, T, F. Tim's comment eliminates the first of the those, at the same that Ethan's exclamation eliminates the second, leaving the third. We eliminate the others since in those scenarios, given the fact they know whether their corresponding person is credible, they would otherwise be able to tell the truth values of all the others as well.

9. Pick an integer n between 1 and 10. If no other team picks the same number, we'll give you $\frac{n}{10}$ points.

Answer: varies

Solution: We found that most teams either went for "it all" by choosing 10, or were much more conservative with 5's and 6's. Hardly any teams picked 9 – none did at the University of Florida test site.

10. Many quantities in high-school mathematics are left undefined. Propose a definition or value for the following expressions and justify your response for each. We'll give you $\frac{1}{5}$ points for each reasonable argument.

(i) $(.5)!$ (ii) $\infty \cdot 0$ (iii) 0^0 (iv) $\prod_{x \in \emptyset} x$ (v) $1 - 1 + 1 - 1 + \cdots$

Answer: varies

Solution: Grader discretion was used. For (i), the "most correct" answer uses something of the gamma function, discussed at `http://en.wikipedia.org/wiki/Gamma_function`. For (ii), (iii), and (v) graders accepted most logical arguments; 0, 1, and $\frac{1}{2}$, respectively were common answers. Usually the "empty product" in (iv) is defined as 1, but some teams had other interesting ideas.

11. On the back of your answer sheet, write the "coolest" math question you know, and include the solution. If the graders like your question the most, then you'll get a point. (With your permission, we might include your question on the Mixer next year!)

Answer: various

Solution: You'll find out next year if we use your question!