. . . . Mathathon Round 1 (2 points each) . . . .

1. If this mathathon has 7 rounds of 3 problems each, how many problems does it have in total? (Not a trick!)

Answer: 21

Solution: $7 \cdot 3 = 21$
(Difficulty: 1)

2. Five people, named A, B, C, D, and E, are standing in line. If they randomly rearrange themselves, what's the probability that nobody is more than one spot away from where they started?

Answer: $\frac{1}{15}$

Solution: Either nobody can swap (1 way), one person can swap (4 ways), or two people can swap (3 ways). So $\frac{8}{5!} = \frac{1}{15}$.
(Difficulty: 2)

3. At Barrios's absurdly priced fish and chip shop, one fish is worth \$13, one chip is worth \$5. What is the largest amount of money a customer can enter with, and not be able to spend it all on fish and chips?

Answer: \$47

Solution: Apply the Chicken McNugget theorem.
(Difficulty: 1)

. . . . Mathathon Round 2 (3 points each) . . . .

4. If there are 15 points in 4-dimensional space, what is the maximum number of hyperplanes that these points determine?

Answer: 1365

Solution: In 3d space, 3 points determine a plane, and analogously, in 4d space, 4 points determine a hyperplane. At a maximum, every group of 4 determines a unique hyperplane, and so there are $\binom{14}{4} = 1365$ hyperplanes.
(Difficulty: 2)

5. Consider all possible values of

$$\frac{z_1 - z_2}{z_2 - z_3} \cdot \frac{z_1 - z_4}{z_2 - z_4}$$

for any distinct complex numbers z_1, z_2, z_3, and z_4. How many complex numbers cannot be achieved?

Answer: 1

Solution: Note that the transformation $z \mapsto \frac{z_1 - z_2}{z_2 - z_3} \cdot \frac{z_1 - z}{z_2 - z} = \frac{-\lambda z + \lambda z_1}{-z + z_2}$, with λ equal to the first constant factor, is a fractional linear transformation with nonzero determinant (since $z_2 \neq z_1$ as the complex numbers are distinct). Therefore the mapping is bijective. Since $f(z_2) = 0$ this cannot be achieved for any other values for z_4, and these are the only one (besides infinity, but we are not dealing with the extended complex numbers).
(Difficulty: 3)

6. For each positive integer n, let $S(n)$ denote the number of positive integers $k \leq n$ such that $\gcd(k, n) = \gcd(k + 1, n) = 1$. Find $S(2015)$.

Answer: 957

Solution: First, observe that $2015 = 5 \cdot 13 \cdot 31$. By the Chinese Remainder Theorem, a positive integer $k \leq n$ is uniquely determined by its residue classes modulo 5, 13, and 31. A positive integer $k \leq n$ satisfies $\gcd(k, n) = \gcd(k + 1, n) = 1$ if and only if k and $k + 1$ are both relatively prime to 5, 13, and 31. Therefore, there are $5 - 2 = 3$ possible residue classes for k

modulo 5, $13 - 2 = 11$ possible residue classes for k modulo 13, and $31 - 2 = 29$ possible residue classes for k modulo 31. Hence, $S(2015) = 3 \cdot 11 \cdot 29 = 957$.
NOTE: The function S is known as a Schemmel totient function, and is more commonly denoted S_2. In general, for a positive integer r, $S_r(n)$ is defined to be the number of positive integers $k \leq n$ such that $\gcd(k+i, n) = 1$ for all $i \in \{0, 1, \ldots, r-1\}$. The Schemmel totient functions S_r are multiplicative arithmetic functions (meaning $S_r(mn) = S_r(m)S_r(n)$ whenever $\gcd(m, n) = 1$) that satisfy

$$S_r(p^\alpha) = \begin{cases} 0, & \text{if } p \leq r; \\ p^{\alpha-1}(p-r), & \text{if } p > r \end{cases}$$

for all primes p and positive integers α.
(Difficulty: 2)

.... Mathathon Round 3 (4 points each) ....

7. Let $P_1, P_2, \ldots, P_{2015}$ be 2015 distinct points in the plane. For any $i, j \in \{1, 2, \ldots, 2015\}$, connect P_i and P_j with a line segment if and only if $\gcd(i - j, 2015) = 1$. Define a clique to be a set of points such that any two points in the clique are connected with a line segment. Let ω be the unique positive integer such that there exists a clique with ω elements and such that there does not exist a clique with $\omega + 1$ elements. Find ω.

Answer: 5

Solution: In any set of 6 positive integers, two of them must be congruent modulo 5. Therefore, in any set of 6 points, two of them cannot be connected. Hence, there are no cliques with 6 elements. On the other hand, P_1, P_2, P_3, P_4, and P_5 form a clique of order 5.
(Difficulty: 2)

8. A Chinese restaurant has many boxes of food. The manager notices that

- He can divide the boxes into groups of M where M is 19, 20, or 21.
- There are exactly 3 integers x less than 16 such that grouping the boxes into groups of x leaves 3 boxes left over.

Find the smallest possible number of boxes of food.

Answer: 39900

Solution: The only integers which can divide N, the number of boxes, and have a remainder of 3 are 8, 9, 11, and 13. (Everything else evenly divides). But since $20|N$, it must be 0 or 4 mod 8. Now we use the Chinese Remainder Theorem mod 9, 11, and 13 of $2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 19 = 7980$ to see that we must scale by 5 to achieve the other condition. So the answer is 39900.
(Difficulty: 3)

9. If $f(x) = x|x| + 2$, then compute $\displaystyle\prod_{k=-1000}^{1000} f^{-1}(f(k) + f(-k) + f^{-1}(k))$.

Answer: 0

Solution: Consider $f(x)$ as a piecewise function, and first consider the case where x is positive. Then $y = x^2 + 2$. The inverse of this is $y = \sqrt{x-2}$, and this function is not $\pm$ since the range is for positive x. Then when x is negative, $y = -x^2 + 2$. The inverse is then $y = -\sqrt{2-x}$ since the range is negative. This inverse function equals 0 whenever $x = 2$. Also $f(k) + f(-k) = 4$. So when $k = -2$, $f^{-1}(k) = -2$, and $f^{-1}(4-2) = f^{-1}(2) = 0$.
(Difficulty: 2)

. . . . Mathathon Round 4 (6 points each) . . . .

10. Let ABC be a triangle with $AB = 13$, $BC = 20$, $CA = 21$. Let $ABDE$, $BCFG$, and $CAHI$ be squares built on sides AB, BC, and CA, respectively such that these squares are outside of ABC. Find the area of $DEHIFG$.

Answer: 1514

Solution: By Herons Formula, $[ABC] = (s(sa)(sb)(sc))^{1/2} = (27 * 14 * 7 * 6)^(1/2) = 126$. Consider $DBG.[DBG] = \frac{1}{2} * DB * BG * \sin(\angle DBG) = \frac{1}{2} * AB * BC * \sin(180\angle DBG) = \frac{1}{2} * AB * BC * \sin(\angle ABC) = [ABC] = 126(since 360 = \angle DBG + \angle ABD + \angle ABC + \angle GBC = \angle DBG + \angle ABC + 90 + 90). Similarly, [FCI] = [HAE] = 126$. $[DEFGHI] = [ABC] + [DBG] + [FCI] + [HAE] + [ABDE] + [BCFG] + [CAHI] = 4[ABC] + [ABDE] + [BCFG] + [CAHI] = 4 \cdot 126 + 169 + 400 + 441 = 1514$.
(Difficulty: 3)

11. What is the sum of all of the distinct prime factors of $7783 = 6^5 + 6 + 1$?

Answer: 224

Solution: Note that

$$\begin{aligned} 6^5 + 6 + 1 &= 6^5 6^2 + 6^2 + 6 + 1 \\ &= 6^2(6^3 1) + 6^2 + 6 + 1 \\ &= (6^2 + 6 + 1)(6^2(61) + 1) \\ &= 43 \cdot 181. \end{aligned}$$

Since 43 and 181 are both primes, the sum of the prime factors of 7783 is 224.
(Difficulty: 3)

12. Consider polyhedron $ABCDE$, where $ABCD$ is a regular tetrahedron and $BCDE$ is a regular tetrahedron. An ant starts at point A. Every time the ant moves, it walks from its current point to an adjacent point. The ant has an equal probability of moving to each adjacent point. After 6 moves, what is the probability the ant is back at point A?

Answer: 11/64

Solution: Define a_n to be the probability the ant is at point A after n moves. Define bcd_n to be the probability the ant is at points B, C, or D after n moves. Define e_n to be the probability the ant is at point E after n moves. If the ant is at points B, C, or D, the probability that the ant moves to point A is . Similarly, if the ant is at points B, C, or D, the probability that the ant moves to point E is $\frac{1}{4}$. Therefore, $a_n = \frac{1}{4} * bcd_{n-1}$ and $e_n = \frac{1}{4} * bcd_(n-1)$. Given that $a_1 = e_1 = 0$ and $bcd_1 = 1, a_6 = 11/64$.
(Difficulty: 2)

. . . . Mathathon Round 5 (7 points each) . . . .

13. You have a 26×26 grid of squares. Color each randomly with red, yellow, or blue. What is the expected number (to the nearest integer) of 2×2 squares that are entirely red?

Answer: 8

Solution: 625/81 There are 25^2 two by two blocks. Any block has a $1/3^4$ chance of being all red. Then use linearity of expectations
(Difficulty: 3)

14. Four snakes are boarding a plane with four seats. Each snake has been assigned to a different seat. The first snake sits in the wrong seat. Any subsequent snake will sit in their assigned seat if possible, if not, they will choose a random seat. What is the expected number of snakes who sit in their correct seats?

Answer: $\frac{14}{9}$

Solution: The basic idea is we first find the probability that for $2 \leq k \leq n$, the kth snake doesn't get his correct seat.

Because he has to get kicked out by a guy that's smaller, it's not hard to see that the sequence of kicking-out needs to go $1 \to a_1 \to \cdots \to a_m \to k$ for some increasing sequence $a_1, \ldots, a_m$. This sequence (we can ignore everything in between since those peeps get their seats with probability 1) has probability of $1/(n-1) * 1/(n+1-a_1) * ... * 1/(n+1-a_m)$.
Summing this over all possible subsets $\{a_1, ..., a_m\}$ in $\{2, ..., k\}$ is actually just equal to $1/(n-1) * \Pi_{j=2}^{k-1}(1 + 1/(n+1-j))$. Then telescoping gives us $n/(n-1) * (1/(n+2-k))$.
Now summing over $k = 2, \ldots, n$ yields $n/(n-1) * (H_n - 1)$. Adding in the case for $k = 1$ (which is just 1) gives us $(n * H_n - 1)/(n-1)$.
However, since this is the expected number of incorrect seats, we simply subtract from n to get the expected number of correct seats, which is $n - (nH_n - 1)/(n-1)$.
(Difficulty: 4)

15. Let $n \geq 1$ be an integer and $a > 0$ a real number In terms of n, find the number of solutions $(x_1, ...x_n)$ of the equation

$$\sum_{i=1}^{n}(x_i^2 + (a - x_i)^2) = na^2$$

such that x_i belongs to the interval $[0, a]$, for $i = 1, 2, ..., n$.

Answer: 2^n

Solution:

$$na^2 = \sum_{i=1}^{n}(x_i^2 + (a - x_i)^2)$$

$$= 2\sum_{i=1}^{n}(x_i^2 + na^2 - 2a\sum_{i=1}^{n}(x_i)$$

basecally you reduce down to $\sum((x_i)(x_i - a)) = 0$ so either $x_i = 0$ or $x_i = a$ for each i, for each x there are two choices so there are 2^n solutions.
(Difficulty: 3)

.... Mathathon Round 6 (8 points each) ....

16. All roots of

$$\prod_{n=1}^{25}\sum_{k=0}^{2n}(-1)^k \cdot x^k = 0$$

are written in the form $r(\cos\varphi + i\sin\varphi)$ for $i^2 = -1$, $r > 0$, and $0 \leq \varphi < 2\pi$. What is the smallest positive value of φ in radians?

Answer: $\frac{\pi}{51}$

Solution: Note that the product is

$$(x^2 - x + 1)(x^4 - x^3 + x^2 - x + 1)\cdots(x^{50} - x^{49} + \cdots + 1),$$

which we can multiply by $(x + 1)^25$ to get $(x^3 + 1)(x^5 + 1)\cdots(x^{51} + 1)$. Therefore the root of -1 with the smallest angle is $\frac{\pi}{51}$ (half the angle between roots).
(Difficulty: 3)

17. Find the sum of the distinct real roots of the equation

$$\sqrt[3]{x^2 - 2x + 1} + \sqrt[3]{x^2 - x - 6} = \sqrt[3]{2x^2 - 3x - 5}.$$

Answer: $\frac{7}{2}$

Solution: Let $a = \sqrt[3]{x^2 - 2x - 1}$, $b = \sqrt[3]{x^2 - x - 6}$, and $c = \sqrt[3]{2x^2 - 3x - 5}$. Note that $a^3 + b^3 = c^3$. So $a + b = c \Rightarrow (a + b)^3 = c^3 \Rightarrow a^3 + b^3 + 3ab^2 + 3a^2b = c^3 \Rightarrow 3ab(a + b) = 0$. So $a = 0$, $b = 0$, or $0 = a + b = c$. The only distinct root of a

is 1. The second quadratic has two distinct roots that sum to 1. The roots of c are both real, since $9 + 4 \cdot 10 > 0$ and their sum is $\frac{3}{2}$ by Vieta. So the sum of all the roots is $2 + \frac{3}{2} = \frac{7}{2}$.
(Difficulty: 4)

18. If a and b satisfy the property that $a2^n + b$ is a square for all positive integers n, find all possible value(s) of a.

Answer: 0

Solution: Consider the problem in the traditional mods relevant to squares 2,3, use this to derive constraints on a and b. Deduce that this is a square.
(Difficulty: 3)

.... Mathathon Round 7 (9 points each) ....

19. Compute $(1 - \cot 19°)(1 - \cot 26°)$.

Answer: 2

Solution: Converting to sins and cosins gives $\left(\frac{\sin 19° - \cos 19°}{\sin 19°}\right)\left(\frac{\sin 26° - \cos 26°}{\sin 26°}\right)$. But $\sqrt{2} = \frac{1}{\cos 45°} = \frac{1}{\sin 45°}$, so multiply through by 2 to be able to write the product as

$$2\left(\frac{\sin 19° \cos 45° - \cos 19° \sin 45°}{\sin 19°}\right)\left(\frac{\sin 26° \cos 45° - \cos 26° \sin 45°}{\sin 26°}\right).$$

Then we can use the angle addition/subtraction formulas to get $2\left(\frac{\sin(19° - 45°) \times \sin(26° - 45°)}{\sin 19° \sin 26°}\right) = 2$.
(Difficulty: 5)

20. Consider triangle ABC with $AB = 3$, $BC = 5$, and $\angle ABC = 120$. Let point E any point inside ABC. The minimum of the sum of the squares of the distances from E to the three sides of ABC can be written in the form $\frac{a}{b}$, where a and b are natural numbers such that the greatest common divisor of a and b is 1. Find $a + b$.

Answer: 1007

Solution: By the Law of Cosines, $CA = 7$. $[ABC] = \frac{1}{2} \cdot 3 \cdot 5 \cdot \sin(120°) = \frac{15\sqrt{3}}{4}$. Let x be the distance from E to AB, y be the distance from E to BC, and z be the distance from E to CA. Then $[ABC] = \frac{1}{2}(3x + 5y + 7z) = \frac{15\sqrt{3}}{4}$, so $(3x+5y+7z)^2 = 225 * 3/4 = 675/4$. By the Cauchy-Schwarz Inequality, $(3x+5y+7z)^2 \le (3^2+5^2+7^2)(x^2+y^2+z^2), sox^2 + y^2 + z^2 \ge 675/(4 \cdot 83) = 675/332$. Therefore, the answer is 1007.
(Difficulty: 4)

21. Let $m \neq 1$ be a square-free number (not necessarily positive) we denote $\mathbb{Q}(\sqrt{m})$ to be the set of all $a + b\sqrt{m}$ where a and b are rational numbers. Now for a fixed m, let S be the set of all numbers x in $\mathbb{Q}(\sqrt{m})$ such that x is a solution to a polynomial of the form: $x^n + a_1 x^{n-1} + ... + a_n = 0$, where $a_0, \ldots, a_n$ are integers. For many integers m, $S = \mathbb{Z}[\sqrt{m}] = \{a + b\sqrt{m}\}$ where a and b are integers. Give a classification of the integers other than $m = 1$ for which this is not true. (Hint: It is true for $m = -1$ and 2.)

Answer: $m \equiv 1 \pmod 4$ **(not required: except** $m = 1$**)**

Solution: Consider the polynomial with roots $a + b\sqrt{m}$ and $a - b\sqrt{m}$ since irrational roots come in conjugate pairs. Consider when this polynomial has integer coefficients.
(Difficulty: 5)