Mathathon Round 1 (2 points each)

1. Elaine creates a sequence of positive integers $\{s_n\}$. She starts with $s_1 = 2018$. For $n \geq 2$, she sets $s_n = \frac{1}{2}s_{n-1}$ if s_{n-1} is even and $s_n = s_{n-1} + 1$ if s_{n-1} is odd. Find the smallest positive integer n such that $s_n = 1$, or submit "0" as your answer if no such n exists.

2. Alice rolls a fair six-sided die with the numbers 1 through 6, and Bob rolls a fair eight-sided die with the numbers 1 through 8. Alice wins if her number divides Bob's number, and Bob wins otherwise. What is the probability that Alice wins?

3. Four circles each of radius $\frac{1}{4}$ are centered at the points $(\pm\frac{1}{4}, \pm\frac{1}{4})$, and ther exists a fifth circle is externally tangent to these four circles. What is the radius of this fifth circle?

Mathathon Round 2 (3 points each)

4. If Anna rows at a constant speed, it takes her two hours to row her boat up the river (which flows at a constant rate) to Bob's house and thirty minutes to row back home. How many minutes would it take Anna to row to Bob's house if the river were to stop flowing?

5. Let $a_1 = 2018$, and for $n \geq 2$ define $a_n = 2018^{a_{n-1}}$. What is the ones digit of a_{2018}?

6. We can write $(x + 35)^n = \sum_{i=0}^{n} c_i x^i$ for some positive integer n and real numbers c_i. If $c_0 = c_2$, what is n?

Mathathon Round 3 (4 points each)

7. How many positive integers are factors of 12! but not of $(7!)^2$?

8. How many ordered pairs $(f(x), g(x))$ of polynomials of degree at least 1 with integer coefficients satisfy

$$f(x)g(x) = 50x^6 - 3200?$$

9. On a math test, Alice, Bob, and Carol are each equally likely to receive any integer score between 1 and 10 (inclusive). What is the probability that the average of their three scores is an integer?

Mathathon Round 4 (6 points each)

10. Find the largest positive integer N such that $(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)$ is divisible by N for all choices of positive integers $a > b > c > d > e$.

11. Let $ABCDE$ be a square pyramid with $ABCD$ a square and E the apex of the pyramid. Each side length of $ABCDE$ is 6. Let $ABCDD'C'B'A'$ be a cube, where AA', BB', CC', DD' are edges of the cube. Andy the ant is on the surface of $EABCDD'C'B'A'$ at the center of triangle ABE (call this point G) and wants to crawl on the surface of the cube to D'. What is the length the shortest path from G to D'? Write your answer in the form $\sqrt{a + b\sqrt{3}}$, where a and b are positive integers.

12. A *six-digit palindrome* is a positive integer between $100,000$ and $999,999$ (inclusive) which is the same read forwards and backwards in base ten. How many composite six-digit palindromes are there?

. . . . Mathathon Round 5 (7 points each) . . . .

13. Circles ω_1, ω_2, and ω_3 have radii 8, 5, and 5, respectively, and each is externally tangent to the other two. Circle ω_4 is internally tangent to ω_1, ω_2, and ω_3, and circle ω_5 is externally tangent to the same three circles. Find the product of the radii of ω_4 and ω_5.

14. Pythagoras has a regular pentagon with area 1. He connects each pair of non-adjacent vertices with a line segment, which divides the pentagon into ten triangular regions and one pentagonal region. He colors in all of the obtuse triangles. He then repeats this process using the smaller pentagon. If he continues this process an infinite number of times, what is the total area that he colors in? Please rationalize the denominator of your answer.

15. Maisy arranges 61 ordinary yellow tennis balls and 3 special purple tennis balls into a $4 \times 4 \times 4$ cube. (All tennis balls are the same size.) If she chooses the tennis balls' positions in the cube randomly, what is the probability that no two purple tennis balls are touching?

. . . . Mathathon Round 6 (8 points each) . . . .

16. Points A, B, C, and D lie on a line (in that order), and $\triangle BCE$ is isosceles with $\overline{BE} = \overline{CE}$. Furthermore, F lies on $\overline{BE}$ and G lies on $\overline{CE}$ such that $\triangle BFD$ and $\triangle CGA$ are both congruent to $\triangle BCE$. Let H be the intersection of $\overline{DF}$ and $\overline{AG}$, and let I be the intersection of $\overline{BE}$ and $\overline{AG}$. If $m\angle BCE = \arcsin(\frac{12}{13})$, what is $\frac{HI}{FI}$?

17. Three states are said to form a tri-state area if each state borders the other two. What is the maximum possible number of tri-state areas in a country with fifty states? Note that states must be contiguous and that states touching only at "corners" do not count as bordering.

18. Let a, b, c, d, and e be integers satisfying

$$2(\sqrt[3]{2})^2 + 2\sqrt[3]{2}a + 2b + (\sqrt[3]{2})^2 c + \sqrt[3]{2}d + e = 0$$

and

$$25\sqrt{5}i + 25a - 5\sqrt{5}ib - 5c + \sqrt{5}id + e = 0$$

where $i = \sqrt{-1}$. Find $|a + b + c + d + e|$.

. . . . Mathathon Round 7 (9 points each) . . . .

19. What is the greatest number of regions that 100 ellipses can divide the plane into? Include the unbounded region.

20. All of the faces of the convex polyhedron $\mathcal{P}$ are congruent isosceles (but NOT equilateral) triangles that meet in such a way that each vertex of the polyhedron is the meeting point of either ten base angles of the faces or three vertex angles of the faces. (An isosceles triangle has two base angles and one vertex angle.) Find the sum of the numbers of faces, edges, and vertices of $\mathcal{P}$.

21. Find the number of ordered 2018-tuples of integers $(x_1, x_2, \ldots x_{2018})$, where each integer is between -2018^2 and 2018^2 (inclusive), satisfying

$$6(1x_1 + 2x_2 + \cdots + 2018x_{2018})^2 \geq (2018)(2019)(4037)(x_1^2 + x_2^2 + \cdots + x_{2018}^2).$$