Mathathon Round 1 (2 points each)

1. If this mathathon has 7 rounds of 3 problems each, how many problems does it have in total? (Not a trick!)

2. Five people, named A, B, C, D, and E, are standing in line. If they randomly rearrange themselves, what's the probability that nobody is more than one spot away from where they started?

3. At Barrios's absurdly priced fish and chip shop, one fish is worth \$13, one chip is worth \$5. What is the largest integer dollar amount of money a customer can enter with, and not be able to spend it all on fish and chips?

Mathathon Round 2 (3 points each)

4. If there are 15 points in 4-dimensional space, what is the maximum number of hyperplanes that these points determine?

5. Consider all possible values of

$$\frac{z_1 - z_2}{z_2 - z_3} \cdot \frac{z_1 - z_4}{z_2 - z_4}$$

for any distinct complex numbers z_1, z_2, z_3, and z_4. How many complex numbers cannot be achieved?

6. For each positive integer n, let $S(n)$ denote the number of positive integers $k \leq n$ such that $\gcd(k, n) = \gcd(k+1, n) = 1$. Find $S(2015)$.

Mathathon Round 3 (4 points each)

7. Let $P_1, P_2, \ldots, P_{2015}$ be 2015 distinct points in the plane. For any $i, j \in \{1, 2, \ldots, 2015\}$, connect P_i and P_j with a line segment if and only if $\gcd(i-j, 2015) = 1$. Define a clique to be a set of points such that any two points in the clique are connected with a line segment. Let ω be the unique positive integer such that there exists a clique with ω elements and such that there does not exist a clique with $\omega + 1$ elements. Find ω.

8. A Chinese restaurant has many boxes of food. The manager notices that

- He can divide the boxes into groups of M where M is 19, 20, or 21.
- There are exactly 3 integers x less than 16 such that grouping the boxes into groups of x leaves 3 boxes left over.

Find the smallest possible number of boxes of food.

9. If $f(x) = x|x| + 2$, then compute $\displaystyle\prod_{k=-1000}^{1000} f^{-1}(f(k) + f(-k) + f^{-1}(k))$.

. . . . . Mathathon Round 4 (6 points each) . . . .

10. Let ABC be a triangle with $AB = 13$, $BC = 20$, $CA = 21$. Let $ABDE$, $BCFG$, and $CAHI$ be squares built on sides AB, BC, and CA, respectively such that these squares are outside of ABC. Find the area of $DEHIFG$.

11. What is the sum of all of the distinct prime factors of $7783 = 6^5 + 6 + 1$?

12. Consider polyhedron $ABCDE$, where $ABCD$ is a regular tetrahedron and $BCDE$ is a regular tetrahedron. An ant starts at point A. Every time the ant moves, it walks from its current point to an adjacent point. The ant has an equal probability of moving to each adjacent point. After 6 moves, what is the probability the ant is back at point A?

. . . . . Mathathon Round 5 (7 points each) . . . .

13. You have a 26×26 grid of squares. Color each randomly with red, yellow, or blue. What is the expected number (to the nearest integer) of 2×2 squares that are entirely red?

14. Four snakes are boarding a plane with four seats. Each snake has been assigned to a different seat. The first snake sits in the wrong seat. Any subsequent snake will sit in their assigned seat if vacant, if not, they will choose a random seat that is available. What is the expected number of snakes who sit in their correct seats?

15. Let $n \geq 1$ be an integer and $a > 0$ a real number. In terms of n, find the number of solutions $(x_1, \ldots x_n)$ of the equation

$$\sum_{i=1}^{n}(x_i^2 + (a - x_i)^2) = na^2$$

such that x_i belongs to the interval $[0, a]$, for $i = 1, 2, \ldots, n$.

. . . . . Mathathon Round 6 (8 points each) . . . .

16. All roots of

$$\prod_{n=1}^{25}\sum_{k=0}^{2n}(-1)^k \cdot x^k = 0$$

are written in the form $r(\cos\varphi + i\sin\varphi)$ for $i^2 = -1$, $r > 0$, and $0 \leq \varphi < 2\pi$. What is the smallest positive value of φ in radians?

17. Find the sum of the distinct real roots of the equation

$$\sqrt[3]{x^2 - 2x + 1} + \sqrt[3]{x^2 - x - 6} = \sqrt[3]{2x^2 - 3x - 5}.$$

18. If a and b satisfy the property that $a2^n + b$ is a square for all positive integers n, find all possible value(s) of a.

. . . . . Mathathon Round 7 (9 points each) . . . .

19. Compute $(1 - \cot 19^\circ)(1 - \cot 26^\circ)$.

20. Consider triangle ABC with $AB = 3$, $BC = 5$, and $\angle ABC = 120$. Let point E be any point inside ABC. The minimum of the sum of the squares of the distances from E to the three sides of ABC can be written in the form $\frac{a}{b}$, where a and b are natural numbers such that the greatest common divisor of a and b is 1. Find $a + b$.

21. Let $m \neq 1$ be a square-free number (an integer – possibly negative – such that no square divides m). We denote $\mathbb{Q}(\sqrt{m})$ to be the set of all $a + b\sqrt{m}$ where a and b are rational numbers. Now for a fixed m, let S be the set of all numbers x in $\mathbb{Q}(\sqrt{m})$ such that x is a solution to a polynomial of the form: $x^n + a_1x^{n-1} + ... + a_n = 0$, where $a_0, \ldots, a_n$ are integers. For many integers m, $S = \mathbb{Z}[\sqrt{m}] = \{a + b\sqrt{m}\}$ where a and b are integers. Give a classification of the integers for which this is not true. (Hint: It is true for $m = -1$ and 2.)