. . . . Mathathon Round 1 (2 points each) . . . .

1. Elaine creates a sequence of positive integers $\{s_n\}$. She starts with $s_1 = 2018$. For $n \geq 2$, she sets $s_n = \frac{1}{2}s_{n-1}$ if s_{n-1} is even and $s_n = s_{n-1} + 1$ if s_{n-1} is odd. Find the smallest positive integer n such that $s_n = 1$, or submit "0" as your answer if no such n exists.

Answer: 16

Solution: We may compute $s_1 = 2018$, $s_2 = 1009$, $s_3 = 1010$, $s_4 = 505$, $s_5 = 506$, $s_6 = 253$, $s_7 = 254$, $s_8 = 127$, $s_9 = 128$, $s_{10} = 64$, $s_{11} = 32$, $s_{12} = 16$, $s_{13} = 8$, $s_{14} = 4$, $s_{15} = 2$, $s_{16} = 1$.

2. Alice rolls a fair six-sided die with the numbers 1 through 6, and Bob rolls a fair eight-sided die with the numbers 1 through 8. Alice wins if her number divides Bob's number, and Bob wins otherwise. What is the probability that Alice wins?

Answer: $\frac{3}{8}$

Solution: There are 48 possibilities for the two rolls. Alice wins if: she rolls 1 and Bob rolls anything; or she rolls 2 and Bob rolls an even number; or she rolls 3 and Bob rolls 3 or 6; or she rolls 4 and Bob rolls 4 or 8; or both roll 5 or both roll 6. In total, there are $8 + 4 + 2 + 2 + 1 + 1 = 18$ ways for Alice to win. And $\frac{18}{48} = \frac{3}{8}$.

3. Four circles each of radius $\frac{1}{4}$ are centered at the points $(\pm\frac{1}{4}, \pm\frac{1}{4})$, and there exists a fifth circle is externally tangent to these four circles. What is the radius of this fifth circle?

Answer: $\frac{\sqrt{2}-1}{4}$

Solution: Let r be the radius of the fifth circle. The length of the segment from $(-\frac{1}{4}, -\frac{1}{4},)$ to $(\frac{1}{4}, \frac{1}{4},)$ is $\frac{\sqrt{2}}{2} = 2 \cdot \frac{1}{4} + 2r$ and $r = \frac{\sqrt{2}-1}{4}$.

. . . . Mathathon Round 2 (3 points each) . . . .

4. If Anna rows at a constant speed, it takes her two hours to row her boat up the river (which flows at a constant rate) to Bob's house and thirty minutes to row back home. How many minutes would it take Anna to row to Bob's house if the river were to stop flowing?

Answer: 48

Solution: Without loss of generality, let Anna live 4 miles away from Bob. On her way upstream, she is moving at 2 mph, and on her way home, she is moving at 8 mph. Thus she must row at 5 mph, and the river flows at 3 mph. Thus to go the 4 miles from her house to Bob's house, would take $\frac{4}{5}$ hours i.e. 48 minutes.

5. Let $a_1 = 2018$, and for $n \geq 2$ define $a_n = 2018^{a_{n-1}}$. What is the ones digit of a_{2018}?

Answer: 6

Solution: We only care about the value of a_{2018} modulo 10. Note that 8 has order 4 modulo 10 ($8 \to 4 \to 2 \to 6 \to 8$). The exponent of 2018 in a_{2018} is 0 modulo 4, so its last digit is the last digit of 8^4, which is 6.

6. We can write $(x + 35)^n = \sum_{i=0}^{n} c_i x^i$ for some positive integer n and real numbers c_i. If $c_0 = c_2$, what is n?

Answer: 50

Solution: $c_0 = 35^n$ and $c_2 = \binom{n}{2}35^{n-2}$. So $c_0 = c_2 \to \binom{n}{2} = 35^2 \to \frac{(n)(n-1)}{2} = 35^2 \to (n)(n-1) = (50)(49) \to n = 50$. (Alternatively, use the quadratic formula for the last step.)

. . . . Mathathon Round 3 (4 points each) . . . .

7. How many positive integers are factors of 12! but not of $(7!)^2$?

Answer: 522

Solution: Solution 1: Compute $12! = 12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2 = 2^{10}\cdot3^5\cdot5^2\cdot7\cdot11$ and $(7!)^2 = (7\cdot6\cdot5\cdot4\cdot3\cdot2)^2 = (2^4\cdot3^2\cdot5\cdot7)^2 = 2^8\cdot3^4\cdot5^2\cdot7^2$. Let A be the set of factors of 12!, and let B be the set of factors of $(7!)^2$. Clearly, $|A| = (11)(6)(3)(2)(2) = 792$ and $|B| = (9)(5)(3)(3) = 405$. Since $\gcd(12!, (7!)^2) = 2^8\cdot3^4\cdot5^2\cdot7$, we get $|A\cap B| = (9)(5)(3)(2) = 270$. The elements of $B\setminus A$ are the factors of $(7!)^2$ with both 7's, so $|B\setminus A| = (9)(5)(3) = 135$. Finally, we may compute $792+405 = 2(270)+135+|A\setminus B| \longrightarrow |A\setminus B| = 522$. Solution 2: Use inclusion-exclusion. An element of $A\setminus B$ must have at least one of 2^9, 3^5, and 11 as a factor, so $|A\setminus B| = (792)(\frac{2}{11}+\frac{1}{6}+\frac{1}{2}-(\frac{2}{11}\cdot\frac{1}{6}+\frac{2}{11}\cdot\frac{1}{2}+\frac{1}{6}\cdot\frac{1}{2})+\frac{2}{11}\cdot\frac{1}{6}\cdot\frac{1}{2}) = (792)\frac{29}{44} = 522$.

8. How many ordered pairs $(f(x), g(x))$ of polynomials of degree at least 1 with integer coefficients satisfy

$$f(x)g(x) = 50x^6 - 3200?$$

Answer: 180

Solution: Write $50x^6-3200 = (50)(x^6-2^6) = (50)(x^3+2^3)(x^3-2^3) = (2\cdot5^2)(x+2)(x^2-2x+4)(x-2)(x^2+2x+4)$. Ignoring the constants for now, there are $2^4-2 = 14$ ways to "build" $f(x)$ out of these factors (we subtracted off the cases where $f(x)$ or $g(x)$ is constant). There are $(1+1)(2+1) = 6$ factors of 50, so there are 6 ways to distribute the 50 between $f(x)$ and $g(x)$. Since we could also multiply both $f(x)$ and $g(x)$ by -1, the answer is $(2)(14)(6) = 168$.

9. On a math test, Alice, Bob, and Carol are each equally likely to receive any integer score between 1 and 10 (inclusive). What is the probability that the average of their three scores is an integer?

Answer: 0.334

Solution: The average score is an integer exactly when the three scores sum to a multiple of 3. It is easier to solve for when the score goes from 1 to $3n+1$ and plug in $n=3$ at the end. The three scores can sum to a multiple of 3 if all three scores are the same modulo 3 or all different modulo 3. If they are all the same, the number of possibilities is $n^3+n^3+(n+1)^3$, and if they are all different, the number of possibilities is $(3!)(n^2)(n+1)$ for a total of $2n^3+(n^3+3n^2+3n+1)+(6n^3+6n^2) = 9n^3+9n^2+3n+1$. After some algebra: $9n^3+9n^2+3n+1 = \frac{1}{3}(27n^3+27n^2+9n)+1 = \frac{(3n+1)^3-1}{3}+1$. Now plug in $n=3$ to get that the total probability is $\frac{\frac{10^3-1}{3}+1}{10^3} = \frac{\frac{1000-1}{3}+1}{1000} = \frac{333+1}{1000} = \frac{334}{1000} = 0.334$

Mathathon Round 4 (6 points each)

10. Find the largest positive integer N such that $(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)$ is divisible by N for all choices of positive integers $a>b>c>d>e$.

Answer: 288

Solution: Proof: Consider a, b, c, d, and d modulo 2. At least 3 of them must be the same parity by the pigeonhole principle. WLOG, let a, b, and c be the same parity. Then (again by the pigeonhole principle), some two of them differ by a multiple of 4. So we get factors $(4)(2)(2) = 2^4$ from the terms $(a-b)(a-c)(b-c)$. Similarly, the term $(d-e)$ gives us another factor of 2, for a total of 2^5. (If there were 4 or 5 variables of the same parity, that would give us even more factors of 2.) Do the same modulo 3. Here, there have to be at least 2 pairs that are the same modulo 3, so we are guaranteed a factor of 3^2. Thus $(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)$ is always divisible by 288. To see that this is maximal, let $a=5$, $b=4$, $c=3$, $d=2$, and $e=1$, and compute $(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e) = (1)(2)(3)(4)(1)(2)(3)(1)(2)(1) = 288$.

11. Let $ABCDE$ be a square pyramid with $ABCD$ a square and E the apex of the pyramid. Each side length of $ABCDE$ is 6. Let $ABCDD'C'B'A'$ be a cube, where AA', BB', CC', DD' are edges of the cube. Andy the ant is on the surface of $EABCDD'C'B'A'$ at the center of triangle ABE (call this point G) and wants to crawl on the surface of the cube to D'. What is the length the shortest path from G to D'? Write your answer in the form $\sqrt{a+b\sqrt{3}}$, where a and b are positive integers.

Answer: $\sqrt{84+24\sqrt{3}}$

Solution: Let G be Andy's starting point. There are two paths Andy could take. We consider the both cases. Case 1: Andy crawls from ABE across AE to AED across AD to $ADD'A'$. In this case, the shortest distance path from G to D' forms the hypotenuse of right triangle $GA'D'$. GA' has length $6+2\sqrt{3}$ and $A'D'$ has length 6. Applying Pythagorean Theorem yields that $GD'^2 = 84+24\sqrt{3}$. Case 2: Any crawls from ABE across AB to $ABB'A'$ across AA' to $AA'D'D$. Let M be the midpoint of $B'A'$. GD' is the hypotenuse of GMD'. GM has length $6+\sqrt{3}$ and MD has length 9, so $GD'^2 = 120+12\sqrt{3}$. When we compare our two cases' answers, we see that $84+24\sqrt{3} < 120+12\sqrt{3}$, so $\sqrt{84+24\sqrt{3}}$ is our answer.

12. A *six-digit palindrome* is a positive integer between 100,000 and 999,999 (inclusive) which is the same read forwards and backwards in base ten. How many composite six-digit palindromes are there?

Answer: 900

Solution: Any six-digit palindrome is divisible by 11 and hence is composite. There are 9 choices for the first digit and 10 choices each for the second and third digits. In total, there are $9 \cdot 10 \cdot 10 = 900$ such palindromes.

.... Mathathon Round 5 (7 points each) ....

13. Circles ω_1, ω_2, and ω_3 have radii 8, 5, and 5, respectively, and each is externally tangent to the other two. Circle ω_4 is internally tangent to ω_1, ω_2, and ω_3, and circle ω_5 is externally tangent to the same three circles. Find the product of the radii of ω_4 and ω_5.

Answer: $\frac{320}{27}$

Solution: Let the smaller tangent circle have radius r and center D and the larger tangent circle have radius R and center E. Note that ABC forms an isosceles triangle, and drawing the altitude AH perpendicular to BC yields two $5-12-13$ right triangles. (Also note that H is the point of tangency between the circles centered around B and C.) Consider right triangle DHC. By Pythagorean Theorem, we have that $5^2 + (12-8-r)^2 = (5+r)^2$. Solving this equation yields $r = \frac{8}{9}$. Consider right triangle EHC. Let $R = x+5$. By Pythagorean Theorem, we have that $x^2 = 5^2 + (15-x)^2$. Solving this equation yields $R = \frac{40}{3}$. Therefore, the product $Rr = \frac{320}{27}$.

14. Pythagoras has a regular pentagon with area 1. He connects each pair of non-adjacent vertices with a line segment, which divides the pentagon into ten triangular regions and one pentagonal region. He colors in all of the obtuse triangles. He then repeats this process using the smaller pentagon. If he continues this process an infinite number of times, what is the total area that he colors in? Please rationalize the denominator of your answer.

Answer: $\frac{\sqrt{5}-1}{2}$

Solution: It's sufficient to consider what proportion of the "outer" regions is colored at a single stage. Consider adjacent acute and obtuse triangles. The proportion that is colored is

$$\frac{\frac{1+\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2}+1} = \frac{1+\sqrt{5}}{3+\sqrt{5}} = \frac{\sqrt{5}-1}{2}.$$

15. Maisy arranges 61 ordinary yellow tennis balls and 3 special purple tennis balls into a $4 \times 4 \times 4$ cube. (All tennis balls are the same size.) If she chooses the tennis balls' positions in the cube randomly, what is the probability that no two purple tennis balls are touching?

Answer: $\frac{99}{124}$

Solution: There are $\binom{64}{3} = \frac{(64)(63)(62)}{(3)(2)} = (64)(21)(31)$ ways to construct the cube. Use inclusion-exclusion to count the number of ways for at least 2 of the purple tennis balls to be touching. There are 3 "directions" that two purple tennis balls could touch, and in each direction there are $(4^2)(3) = 48$ pairs of adjacent spots. And there are 62 places to put the third purple ball. Factoring in all 3 directions gives $(3)(48)(62) = 8928$ ways. Now we need to subtract off the cases where all 3 balls are touching because we double-counted. Either all 3 are in a straight line, or they form a "L" shape. There are $(3)(4^2)(2) = 96$ ways to do this (3 directions). For the "L" shapes, note that any 2×2 square corresponds to 4 "L" shapes. There are $(3)(4) = 12$ planes the squares could lie in, and in each plane there are $(3)(3) = 9$ possible squares. In total, there are $(4)(12)(9) = 432$ "L" shapes. So the number of arrangements with at least 2 purple balls touching is $8928 - 96 - 432 = 8400$. Then our solution is $1 - \frac{8400}{(64)(21)(31)} = 1 - \frac{25}{124} = \frac{99}{124}$

16. Points A, B, C, and D lie on a line (in that order), and $\triangle BCE$ is isosceles with $\overline{BE} = \overline{CE}$. Furthermore, F lies on $\overline{BE}$ and G lies on $\overline{CE}$ such that $\triangle BFD$ and $\triangle CGA$ are both congruent to $\triangle BCE$. Let H be the intersection of $\overline{DF}$ and $\overline{AG}$, and let I be the intersection of $\overline{BE}$ and $\overline{AG}$. If $m\angle BCE = \arcsin(\frac{12}{13})$, what is $\frac{\overline{HI}}{\overline{FI}}$?

Answer: $\frac{2197}{2380}$

Solution: Let $m\angle BCE = \alpha$. Using vertical angles we can find the angles of the small triangle $\triangle FIH$: $m\angle I = 3\alpha - \pi$. Combining this with $m\angle F = \alpha$ gives $m\angle H = 2\pi - 4\alpha$. By the law of sines, the desired quantity is $\frac{\sin\alpha}{\sin(2\pi-4\alpha)} = \frac{\sin\alpha}{-\sin(4\alpha)}$. Write $\sin(4\alpha) = 2\sin(2\alpha)\cos(2\alpha) = 4\sin(\alpha)\cos(\alpha)(1-2\sin(\alpha)^2)$. We may plug in $\sin(\alpha) = \frac{12}{13}$ and $\cos(\alpha) = \frac{5}{13}$ to get $\frac{\sin\alpha}{-\sin(4\alpha)} = \frac{-1}{(4)(\frac{5}{13})(1-\frac{288}{169})} = \frac{-1}{(\frac{20}{13})(\frac{-119}{169})} = \frac{2197}{2380}$.

17. Three states are said to form a tri-state area if each state borders the other two. What is the maximum possible number of tri-state areas in a country with fifty states? Note that states must be contiguous and that states touching only at "corners" do not count as bordering.

Answer: 96

Solution: Re-cast this problem in terms of graph theory: let each state be a vertex, and draw an edge between two vertices if their states are bordering. This can be any planar graph on 50 vertices. A tri-state area is a triangle in the graph. Clearly, if any face is a square or larger polygon in the graph, then we can create more triangles by tri-angulating that face. So every face is a triangle, and we can conclude that $e = \frac{3}{2}f$. From Euler's formula, we have $v - e + f = 2 \to 50 - \frac{3}{2}f + f = 2 \to 48 = \frac{1}{2}f \to f = 96$. Alternatively, an explicit construction of a graph on 50 vertices with only triangular faces will give the same answer.

18. Let a, b, c, d, and e be integers satisfying

$$2(\sqrt[3]{2})^2 + 2\sqrt[3]{2}a + 2b + (\sqrt[3]{2})^2 c + \sqrt[3]{2}d + e = 0$$

and

$$25\sqrt{5}i + 25a - 5\sqrt{5}ib - 5c + \sqrt{5}id + e = 0$$

where $i = \sqrt{-1}$. Find $|a+b+c+d+e|$.

Answer: 7

Solution: Let $p(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e \in \mathbb{Z}[x] \subset \mathbb{Q}[x]$. We are given that $p(\sqrt[3]{2}) = p(\sqrt{5}i) = 0$. Since $p(x) \in \mathbb{Q}[x]$, we can see that both (x^3-2) and (x^2+5) must divide $p(x)$ (and this is intuitive even without abstract algebra considerations). Thus $(x^3-2)(x^2+5) = x^5 + 5x^3 - 2x^2 - 10$ divides $p(x)$ i.e. this is $p(x)$. So $|a+b+c+d+e| = |0+5-2+0-10| = |-7| = 7$. Alternatively, we can split the first equation into $2 + c = 0$, $2a + d = 0$, and $2b + e = 0$ and the second equation into $25 - 5b + d = 0$ and $25a - 5c + e = 0$. We immediately get $c = -2$. Then substitute $d = -2a$ and $e = -2b$ into the last two equations to get two linear equations in a and b, which one could also solve.

. . . . Mathathon Round 7 (9 points each) . . . .

19. What is the greatest number of regions that 100 ellipses can divide the plane into? Include the unbounded region.

Answer: 19802

Solution: Start with 1 ellipse, which divides the plane into 2 regions. When the n-th ellipse is added, it can intersect each of the $(n-1)$ other ellipses at most 4 times, so it adds at most $4(n-1)$ new regions. Note that this is always possible. So the answer is $2 + 4(1+2+3+\cdots+99) = 2 + 4\frac{(99)(100)}{2} = 2 + 19800 = 19802$.

20. All of the faces of the convex polyhedron $\mathcal{P}$ are congruent isosceles (but NOT equilateral) triangles that meet in such a way that each vertex of the polyhedron is the meeting point of either ten base angles of the faces or three vertex angles of

the faces. (An isosceles triangle has two base angles and one vertex angle.) Find the sum of the numbers of faces, edges, and vertices of $\mathcal{P}$.

Answer: 182

Solution: In one of these triangles, call the two congruent sides the legs, and call the third side the base. Let E_1 denote the total number of bases in $\mathcal{P}$ and E_2 denote the number of legs so that $E = E_1 + E_2$ is the total number of edges of $\mathcal{P}$. Furthermore, suppose there are V_1 vertices where 10 edges meet and V_2 vertices where 3 edges meet, and let $V = V_1 + V_2$ be the total number of vertices. Finally, let F denote the number of faces of $\mathcal{P}$. Each face has exactly one base and each base is contained in exactly two faces, so $E_1 = \frac{F}{2}$. Similarly, each face has two legs and each short edge is contained in exactly two faces, so $E_2 = F$. Now, notice that each vertex counted in V_1 is the meeting point of 5 bases and 5 legs, while every vertex counted in V_2 is the meeting point of 3 legs. Therefore, each base contains exactly two vertices counted in V_1 and each leg contains exactly 1, and this counts each such vertex 10 times, so $V_1 = \frac{2E_1+E_2}{10} = \frac{F}{5}$. Similarly, each leg contains exactly one vertex counted in V_2, and this counts each such vertex 3 times, so $V_2 = \frac{E_2}{3} = \frac{F}{3}$. Therefore, $E = E_1 + E_2 = \frac{3F}{2}$ and $V = V_1 + V_2 = \frac{8F}{15}$. Finally, from Euler's formula for polyhedrons, $V - E + F = 2$, we get $\frac{8F}{15} - \frac{3F}{2} + F = 2 \to \frac{F}{30} = 2 \to F = 60$. Thus $E = \frac{3}{2} \cdot 60 = 90$ and $V = \frac{8}{15} \cdot 60 = 32$, and so $F + E + V = 60 + 90 + 32 = 182$. Note: the solid in question is called the "triakis icosahedron" and can be obtained by replacing every face of an icosahedron with the base of a triangular pyramid of sufficiently small height.

21. Find the number of ordered 2018-tuples of integers $(x_1, x_2, \ldots x_{2018})$, where each integer is between -2018^2 and 2018^2 (inclusive), satisfying

$$6(1x_1 + 2x_2 + \cdots + 2018x_{2018})^2 \geq (2018)(2019)(4037)(x_1^2 + x_2^2 + \cdots + x_{2018}^2).$$

Answer: 4037

Solution: Consider the Cauchy-Schwarz inequality for the vectors $(1, 2, \ldots 2018)$ and $(x_1, \ldots x_{2018})$: $|1x_1 + 2x_2 + \cdots + 2018x_{2018}| \leq \sqrt{1^2 + \cdots + 2018^2}\sqrt{x_1^2 + x_2^2 + \cdots + x_{2018}^2}$ and $(1x_1 + \cdots + 2018x_{2018})^2 \leq \frac{(2018)(2019)(4037)}{6}(x_1^2 + \cdots + x_{2018}^2)$ (by the sum of squares formula) with equality only when the two vectors are parallel. So we must have $(x_1, \ldots x_{2018}) = c(1, 2, \ldots 2018)$ for some integer c. By the bounds imposed on the x_i, we see that we must have $-2018 \leq c \leq 2018$, which gives $2(2018) + 1 = 4037$ solutions.