Math Majors of America Tournament for High Schools
2014 Individual Test Solutions

1. For what value of $x > 0$ does $f(x) = (x-3)^2(x+4)$ achieve the smallest value?

Answer: 3

Solution: f has a double root at 3 and a single root at 4. From this we can sketch the graph and see that $f(3) = 0$ and is never negative when $x > 0$, so $f(3)$ must be its smallest value.

2. There are exactly 29 possible values that can be made using one or more of the 5 distinct coins with values 1, 3, 5, 7, and X. What is the smallest positive integral value for X?

Answer: 17

Solution: There will only be $2^5 - 1 = 31$ possible possible values if the sum of any number of the coins is not the value of different coin. However, $3+5 = 1+7$. There are no other repeats, so we need to preserve that by adding another value. (Note that if we would then only have 29 since $3+5+X = 1+7+X$. Since $9 = 3+5+1$, $10 = 7+3$, $11 = 7+3+1$, $12 = 5+7$, and $13 = 1+5+7$, $14+1 = 3+5+7$, $15 = 3+5+7$, $16 = 1+3+5+7$, so 17 is the first value that will work.

3. Define $\star$ as $x \star y = x - \frac{1}{xy}$. What is the sum of all complex x such that $x \star (x \star 2x) = 2x$?

Answer: 0

Solution: So we have $x \star (x \star 2x) = x \star (x - 1/2x^2) = x - \frac{1}{x(x-1/2x^2)} = 2x \Rightarrow 2x^4 - 3x = 4x^4 \Rightarrow 2x^4 + 3x = 0$, so by Vieta's formulas the sum is 0.

4. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x and let $\lceil x \rceil$ be the least integer greater than or equal to x. Compute the smallest positive value of a for which $\lfloor a \rfloor$, $\lceil a \rceil$, $\lfloor a^2 \rfloor$ is a nonconstant arithmetic sequence.

Answer: $\sqrt{3}$

Solution: Clearly $\lfloor a \rfloor$ and $\lceil a \rceil$ differ by exactly 1. Hence $\lfloor a^2 \rfloor - \lceil a \rceil = 1 \Rightarrow \lfloor a^2 \rfloor = 1 + \lceil a \rceil$. If $\lceil a \rceil = 0$, then a is either 0 or negative. If it equaled 1, then $a < 1$, in which case $a^2 < 1 \Rightarrow \lfloor a^2 \rfloor = 0$. So $\lceil a \rceil = 2$, so we need $\lfloor a^2 \rfloor = 3$ with $1 < a < 2$. We note that $a = \sqrt{3}$ is the smallest value with this property.

5. A right triangle is bounded in a coordinate plane by the lines $x = 0$, $y = 0$, $x = x_{100}$, and $y = f(x)$, where f is a linear function with a negative slope and $f(x_{100}) = 0$. The lines $x = x_1$, $x = x_2$, $\ldots$, $x = x_{99}$ ($x_1 < x_2 < \cdots < x_{100}$) subdivide the triangle into 100 regions of equal area. Compute $\frac{x_{100}}{x_1}$.

Answer: $100 + 30\sqrt{11}$

Solution: Let the height of the triangle be h. Then the total area is $\frac{1}{2}h \cdot x_{100}$. If $h_1 = f(x_1)$, then by similar triangles we have $\frac{x_{100}}{h} = \frac{x_{100}-x_1}{h_1}$. The area of the trapezoid that the the y-axis bounds with the line $x = x_{100}$ is the difference $\frac{1}{2}h \cdot x_{100} - \frac{1}{2}h_1 \cdot (x_{100} - x_1)$, and this must be $\frac{1}{100}$ of the total area. So $hx_{100} - (x_{100} - x_1)h_1 = \frac{hx_{100}}{100}$. Dividing through by h and using the similar triangles relationship, we get $x_{100} - (x_{100} - x_1)\frac{(x_{100}-x_1)}{x_{100}} = \frac{x_{100}}{100}$. Next we divide through by x_{100} to get $1 - (1 - \frac{x_1}{x_{100}})^2 = \frac{1}{100}$. We have $x_{100} > x_1$, so we take the positive square root to get $\frac{x_1}{x_{100}} = 1 - \frac{3\sqrt{11}}{10}$. We are looking for the reciprocal, or $30\sqrt{11} + 100$.

6. There are 10 children in a line to get candy. The pieces of candy are indistinguishable, while the children are not. If there are a total of 390 pieces of candy, how many ways are there to distribute the candy so that the n^{th} child in line receives at least n^2 pieces of candy?

Answer: 2002

Solution: First, give each child his minimum amount of candy (n^2) before distributing the excess. $\sum_{n=1}^{10} n^2 = \frac{(10)(10+1)(2(10)+1)}{6} = 385$. Now there are 5 pieces of candy to distribute among 10 children. Use a "'stars and bars"' method by representing each piece of candy with a star and dividing them with bars. For example, a diagram depicting the situation in which the third child gets two extra candies while the sixth, seventh, and ninth children each get one extra would look like this:

$$||**|||*|*||*|$$

There is a one-to-one correspondence between candy distributions and diagrams. The number of such diagrams can be found by noting that there are 14 places to put either a star or a bar and that we must fill 5 of them with stars. The answer is then $\binom{14}{5} = \frac{14!}{5!9!} = 2002$.

7. Compute $\dfrac{\binom{54}{23}+6\binom{54}{24}+15\binom{54}{25}+15\binom{54}{27}+6\binom{54}{28}+\binom{54}{29}-\binom{60}{29}}{\binom{54}{26}}$

Answer: -20

Solution: Note that we are trying to solve the equation $\binom{54}{23}+6\binom{54}{24}+15\binom{54}{25}+15\binom{54}{27}+6\binom{54}{28}+\binom{54}{29}-x\cdot\binom{54}{26}=\binom{60}{29}$. The right hand side is, almost by definition, the number of ways to choose 29 objects from 60. Divide those 60 things into two groups, A and B, with 54 of the things in A and 6 of the things in B. We can choose 0, 1, 2, 3, 4, 5, or 6 of the things from B (in $\binom{6}{0}$, $\binom{6}{1}$, $\binom{6}{2}$, etc. ways) and the remaining (or $29-j$, where j is the number taken from B) from group A. The left hand side can be thought of as the number of ways to do this, and each term corresponds to one possible value of j.The missing coefficient is $-\binom{6}{3}=-20$

8. Point A lies on the circle centered at O. $\overline{AB}$ is tangent to O, and C is located on the circle so that $m\angle AOC = 120°$ and oriented so that $\angle BAC$ is obtuse. $\overline{BC}$ intersects the circle at D. If $AB = 6$ and $BD = 3$, then compute the radius of the circle.

Answer: $-\sqrt{3}+\sqrt{39}$

Solution: First, using the tangent-secant power theorem, find $\overline{BC}$:

$$\overline{BC} = \frac{\overline{AB}^2}{\overline{BD}}$$

$$\overline{BC} = \frac{6^2}{3}$$

$$\overline{BC} = 12$$

Then, knowing that $\angle AOC$ is $\frac{2\pi}{3}$, that $\angle OAB = \frac{\pi}{2}$ (tangent radius relationship), and that $\overline{AO} = \overline{CO}$ (both radii), it can be determined that $\angle BAC = \frac{2\pi}{3}$:

$$\angle OAC = \pi - \angle AOC - \angle OCA$$

$$\angle OAC = \angle OCA$$

$$\angle OAC = \frac{\pi - \frac{2\pi}{3}}{2}$$

$$\angle OAC = \frac{\pi}{6}$$

$$\angle BAC = \angle OAB + \angle OAC$$

$$\angle BAC = \frac{\pi}{2} + \frac{\pi}{6}$$

$$\angle BAC = \frac{2\pi}{3}$$

Next, find $\overline{AC}$ using the law of cosines:

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2\overline{AB}\,\overline{AC}\cos(\angle BAC)$$

$$144 = 36 + \overline{AC}^2 + 6\overline{AC}$$

$$\overline{AC} = \frac{-6+\sqrt{6^2+4\times 108}}{2}$$

$$\overline{AC} = \frac{-6+2\sqrt{117}}{2}$$

$$\overline{AC} = -3+3\sqrt{13}$$

Finally, bisect $\overline{AC}$ and use a 30-60-90 triangle to find $\overline{OA}$, a radius:

$$\overline{OA} = \frac{2}{\sqrt{3}} \times \frac{\overline{AC}}{2}$$

$$\overline{OA} = -\sqrt{3} + \sqrt{39}$$

9. The center of each face of a regular octahedron (a solid figure with 8 equilateral triangles as faces) with side length one unit is marked, and those points are the vertices of some cube. The center of each face of the cube is marked, and these points are the vertices of an even smaller regular octahedron. What is the volume of the smaller octahedron?

Answer: $\frac{\sqrt{2}}{81}$

Solution: The volume of the larger octahedron is $\frac{1}{3} \cdot \sqrt{2}$ by determining that the "height" is $\sqrt{2}$ (by repeated use of the Pythagorean theorem) and the formula $V = \frac{1}{3} \cdot B \cdot H$. Next we note that the new octahedron has the exact same orientation and shape as the old one, except it is scaled. We find the scaling factor by determining how far from the center of the solid figure the new vertices are (i.e. the perpendicular distance of a side of a square to the center). Let A be a new vertex, B be the old vertex closest to A, and O be the center of the figure. The cube has a vertex on each of the four faces adjacent to B; call one of them D. Then $\overline{BD}$ intersects another edge of the old octahedron at point E. We observe that $\triangle EBO$ is similar to $\triangle DBA$. Since D is the centroid of one of the faces, it divides the median in the ratio $2:1$. Hence $\frac{2}{3} = \frac{DB}{EB} = \frac{AB}{OB} = \frac{OB-OA}{OB} = 1 - \frac{OA}{OB}$. Hence $\frac{OA}{OB} = \frac{1}{3}$. So each side length is $\frac{1}{3}$ the corresponding size of the larger solid. Therefore, the volume is scaled by $\frac{1}{27}$ to give $\frac{\sqrt{2}}{81}$.

10. Compute the greatest positive integer n such that there exists an odd integer a, for which $\dfrac{a^{2^n}-1}{4^{4^4}}$ is not an integer.

Answer: 509

Solution: Note that $4^{4^4} = 2^{512}$. We prove that for any $n \geq 3$, $a^{2^{n-2}} \equiv 1 \pmod{2^n}$. The base case $n = 3$ is trivial. Suppose the statement is true for any $n = k$. Then $a^{2^{k-2}} \equiv 1 \pmod{2^k}$, so there exists an integer b such that $a^{2^{k-2}} - 1 = 2^k b$. Rearranging and then squaring gives $a^{2^{k-2}} = 1 + 2^k b \Rightarrow a^{2^{k-1}} = 1 + 2^{k+1}(b + 2^{k-1}b^2) \equiv 1 \pmod{2^{k+1}}$, which completes the induction. Therefore, in the problem, $n = 512 - 2 = 510$ will make the expression be an integer. In adddition, $a^{2^k} \equiv 1 \pmod{2^512} \Rightarrow 2^{512}|(a^{2^k} - 1) \Rightarrow 2^{512}|(a^{2^k} - 1) \cdot (a^{2^k} + 1) = a^{2^{k+1}} - 1 \Rightarrow a^{2^{k+1}} \equiv 1 \pmod{2^{512}}$, so this will hold for all higher $n \geq 510$.
All that is left to show is that for at least one a, $a^{2^{509}} - 1$ is not divisible by 2^{512}. We prove this in the case $a = 3$. Define $V_2(k)$ to be the highest power of 2 that divides k. We claim that $V_2(3^{2^k} - 1) = k + 2$ for all $k \geq 1$. When $k = 1$, $V_2(3^2 - 1) = v_2(2^3) = 3 = 1 + 2$. Suppose $V_2(3^{2^{k-1}} - 1) = k + 1$ for some $k \geq 2$. By the difference of squares factorization, $3^{2^k} - 1 = (3^{2^{k-1}} - 1)(3^{2^{k-1}} + 1)$, so $V_2(3^{2^k} - 1) = V_2(3^{2^{k-1}} - 1) + V_2(3^{2^{k-1}} + 1) = k + 1 + V_2(3^{2^{k-1}} + 1)$ by the inductive hypothesis. Because $k \geq 2$, $3^{2^{k-1}} + 1 \equiv 2 \pmod{8}$, so $V_2(3^{2^{k-1}} + 1) = 1$. So we have $V_2(3^{2^k} - 1) = k + 2$, completing the induction. Therefore if $n = 509$, the highest power of 2 dividing $3^{2^{509}}$ is 511. Hence the expression is not an integer mod 512.

11. Three identical balls are painted white and black, so that half of each sphere is a white hemisphere, and the other half is a black one. The three balls are placed on a plane surface, each with a random orientation, so that each ball has a point of contact with the other two. What is the probability that at at least one point of contact between two of the balls, both balls are the same color?

Answer: $\frac{95}{108}$

Solution: Note that the centers of the balls are the vertices of an equilateral triangle. Look at the cross section of a ball that lies in the plane of this triangle. The angle between the points of contact, as viewed from the center of the great circle of the ball in this plane, is $\frac{\pi}{3}$. The plane that divides the ball into the two hemispheres must intersect the cross section along a diameter. This line is determined by one point of intersection with the circle. There is a $\frac{2}{3}$ chance that the point is at least $\frac{\pi}{3}$ from one of the vertices, in the appropriate direction. We now calculate the probabiblity that each of the points of contact are different colors, or unmatched. For each ball, there are two states: the two points of contact from the ball are the same color, or the points of contact are different colors. Let S be the first case, and D be the second. All contact points are unmatched only when the configuration is DDD, SSD, or some permutation of those. In particular, all contact points are never unmatched when the configuration is SDD or some permutation. The probability of DDD is $\left(\frac{1}{3}\right)^3$ and SSD is $\left(\frac{2}{3}\right)^2 \cdot \frac{1}{3}$, times $3 = \frac{4}{9}$ to account for the permutation of which ball is in the D state. For the DDD configuration, after the first ball's orientation is fixed, there is a $\left(\frac{1}{2}\right)^2$ chance that the other two are oriented correctly. In the SSD case, fix one of the S balls. Then there is a $\frac{1}{2}$ chance the other S ball is oriented properly (so that the point of contact between the two S balls is unmatched). The D ball also has a $\frac{1}{2}$ chance of being in the proper orientation. Overall the probability that all contact points are unmatched is $\frac{1}{27} \cdot \frac{1}{4} + \frac{4}{9} \cdot \frac{1}{4} = \frac{1}{108} + \frac{12}{108} = \frac{13}{108}$. We are looking for the complement, $1 - \frac{13}{108} = \frac{95}{108}$

12. Define an operation Φ whose input is a real-valued function and output is a real number so that it has the following properties:

- For any two real-valued functions $f(x)$ and $g(x)$, and any real numbers a and b, then

$$\Phi(af(x) + bg(x)) = a\Phi(f(x)) + b\Phi(g(x)).$$

- For any real-valued function $h(x)$, there is a polynomial function $p(x)$ such that

$$\Phi(p(x) \cdot h(x)) = \Phi((h(x))^2).$$

- If some function $m(x)$ is always non-negative, and $\Phi(m(x)) = 0$, then $m(x)$ is always 0.

Let $r(x)$ be a real-valued function with $r(5) = 3$. Let S be the set of all real-valued functions $s(x)$ that satisfy that $\Phi(r(x) \cdot x^n) = \Phi(s(x) \cdot x^{n+1})$. For each s in S, give the value of $s(5)$.

Answer: $\frac{3}{5}$

Solution: Define $s' = s \cdot x$ so that the RHS of the condition is $\Phi(s' \cdot x)$. Consider the function $(r - s')(x)$. Let p be the polynomial as in property 2. Then $\Phi((r - s')^2) = \Phi((r - s') \cdot p) = \phi(r \cdot p) - \phi(s' \cdot p)$ by property 1. Suppose $p = a_0 + a_1 x + \cdots + a_n x^n$. Then the previous expression is equal to $\phi(ra_0) + \Phi(ra_1 x) + \Phi(ra_2 x^2) + \cdots \Phi(ra_n x^n) - \Phi(s' a_0) - \Phi(s' a_1 x) - \cdots - \Phi(s' a_n x^n)$. But by the relationship between s' and r, each element of that sum cancels with another one. So $\Phi((r - s')^2) = 0$, and by the third property, $s' - r = 0$, which means the two functions are identical. So $r(x) = xs(x) \Rightarrow 3 = r(5) = 5s(5)$. So $s(5) = \frac{3}{5}$ is the only possible answer.

We remark that the Φ operator has very similar properties as the integral from calculus. In fact, it can be shown that given any function and any closed interval, there is a sequence of polynomial functions that uniformly approximates the arbitrary function. The limit of the sequence of integrals of the polynomial functions converges to the integral of the original funciton. Therefore, property 2 does not need to be assumed. By the same techniques of problem 12, it can be shown that if for all $n \geq 0$ we have $\int f \cdot x^n \, \mathrm{d}x = \int g \cdot x^n \, \mathrm{d}x$, then $f \equiv g$ (that is, then f and g are the same functions). More surprisingly, if a_n is a sequence of integers, then $\left(\int f \cdot x^{a_n} \, \mathrm{d}x = \int g \cdot x^{a_n} \, \mathrm{d}x \Rightarrow f \equiv g\right)$ if and only if $\sum_{i=1}^{\infty} \frac{1}{a_n}$ diverges.