Girls in Math 2020

Individual Round Solutions

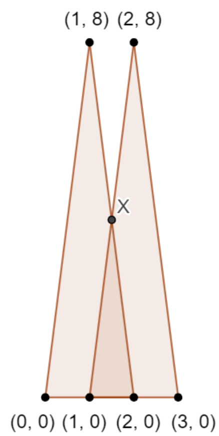
February 1, 2020

Problem 1. If $3a + 1 = b$ and $3b + 1 = 2020$, what is a?

Solution. We see that $3b + 1 = 2020 \implies 3b = 2019$, so $b = 673$; then $3a + 1 = 673 \implies 3a = 672$, so $a = \boxed{224}$. □

Problem 2. Tracy draws two triangles: one with vertices at $(0, 0)$, $(2, 0)$, and $(1, 8)$ and another with vertices at $(1, 0)$, $(3, 0)$, and $(2, 8)$. What is the area of overlap of the two triangles?

Solution. Observe that when we draw a diagram, the area of overlap is a triangle, as shown below:



Two of the vertices of the area of overlap will be $(1, 0)$ and $(2, 0)$. The third, X, can be found by intersecting the lines connecting $(1, 8)$ and $(2, 0)$, and $(1, 0)$ and $(2, 8)$; by symmetry this works out to be $(1.5, 4)$.

Then the triangle has base 1 and height 4, so its area is $\boxed{2}$. □

Problem 3. If p, q, and r are prime numbers such that $p+q+r = 50$, what is the maximum possible value of pqr?

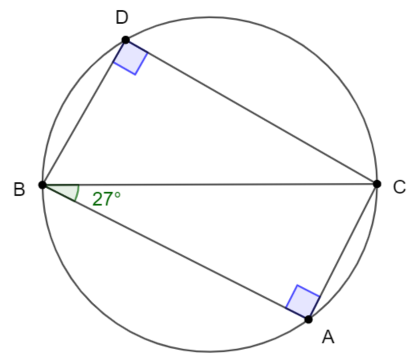
Solution. Note firstly that all primes except for 2 are odd, so therefore if none of p, q, and r equals 2, then $p+q+r$ will be odd. This establishes that at least one of p, q, r must be 2; without loss of generality assume that $r = 2$.

Then $p + q = 48$ and p and q are prime numbers, and we wish to maximize their product. By intuition (or by the AM-GM inequality), we see that we seek to push p and q as close together. Therefore we begin trying pairs of odd numbers: $(23, 25)$ and $(21, 27)$ clearly do not work, but $(19, 29)$ are indeed prime numbers that sum to 48.

Our final answer is $2 \cdot 19 \cdot 29 = \boxed{1102}$. □

Problem 4. Points A, B, C, D lie on a circle of radius 4 such that $BC = 8$, $BD = 4$, and $m\angle ABC = 27°$. If segments $\overline{AB}$ and $\overline{CD}$ do not intersect, what is the value of $m\angle ACD$? (Give your answer in degrees.)

Solution. We begin by drawing a diagram:



Given that $BC = 8$ and that the circle has radius 4, it follows that $\overline{BC}$ is a diameter. Consequently, we see that $\angle BDC = 90°$, and as $BD = 4$ and $BC = 8$, $\triangle BCD$ is a $30 - 60 - 90$ triangle with $\angle BCD = 30°$.

Next, we again use that $\overline{BC}$ is a diameter to conclude that $\angle BCA = 63°$, as a result of right triangle ABC. It follows that $\angle ACD = \angle BCD + \angle BCA = 30° + 63° = \boxed{93°}$. □

Problem 5. Express $\sqrt{14-\sqrt{52}} - \sqrt{14+\sqrt{52}}$ as a rational number.

Solution. Suppose that $a = \sqrt{14-\sqrt{52}} - \sqrt{14+\sqrt{52}}$. To get rid of the annoying nested square roots, we square both sides of the equation, yielding

$$a^2 = (14-\sqrt{52}) - 2\left(\sqrt{14-\sqrt{52}}\right)\left(\sqrt{14+\sqrt{52}}\right) + (14+\sqrt{52}).$$

The $\sqrt{52}$s cancel in the outer terms of the expansion, thankfully, so we need to deal with the term in the middle. We see that

$$\left(\sqrt{14-\sqrt{52}}\right)\left(\sqrt{14+\sqrt{52}}\right) = \sqrt{(14-\sqrt{52})(14+\sqrt{52})} = \sqrt{144} = 12.$$

This makes our equation for a^2 much easier to deal with: plugging in everything and reducing, we get

$$a^2 = 28 - 2(12) = 4.$$

Now note that $\sqrt{14-\sqrt{52}} < \sqrt{14+\sqrt{52}}$, so a is negative. It follows that $a = \boxed{-2}$. □

Problem 6. Let $a_0 = 1$, and let $a_n = 1 + \frac{1}{a_{n-1}}$ for every integer $n \geq 1$. Find the value of the product $a_1 a_2 \cdots a_9$.

Solution. We begin by computing the first few terms of the sequence. Observe that $a_1 = 1 + \frac{1}{1} = 2$; $a_2 = 1 + \frac{1}{2} = \frac{3}{2}$; $a_3 = 1 + \frac{2}{3} = \frac{5}{3}$; $a_4 = 1 + \frac{3}{5} = \frac{8}{5}$. This looks promising: after all,

$$a_1 a_2 a_3 a_4 = 2 \cdot \frac{3}{2} \cdot \frac{5}{3} \cdot \frac{8}{5} = 8,$$

as all the numerators nicely cancel with the denominators.

Additionally, one may observe that all the terms seem to be successive ratios of Fibonacci numbers. At this point, it is possible to proceed by extrapolating the rest of the a_i and assuming they follow this pattern (which will result in the correct answer), but we will provide a rigorous proof.

Let $F_0 = 1, F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ be the Fibonacci Sequence. We will show that $a_i = \frac{F_{i+1}}{F_i}$ for all i.

We proceed by induction. For our base case, observe that $a_0 = 1$ and $\frac{F_1}{F_0} = \frac{1}{1} = 1$. As for our inductive step: suppose that for some k, $a_k = \frac{F_{k+1}}{F_k}$. We will show that $a_{k+1} = \frac{F_{k+2}}{F_{k+1}}$.

We have $a_{k+1} = 1 + \frac{1}{a_k}$, so $a_{k+1} = 1 + \frac{F_k}{F_{k+1}} = \frac{F_{k+1}+F_k}{F_{k+1}}$. But $F_k + F_{k+1} = F_{k+2}$, so $a_{k+1} = \frac{F_{k+2}}{F_{k+1}}$, as desired.

Therefore our product comes out as follows:

$$a_1a_2a_3\dots a_9 = \frac{F_2}{F_1}\cdot\frac{F_3}{F_2}\cdot\frac{F_4}{F_3}\cdot\dots\cdot\frac{F_{10}}{F_9},$$

which nicely reduces to $\frac{F_{10}}{F_1}$. We may compute manually that $F_{10} = 89$, and as $F_1 = 1$ our answer is $\boxed{89}$. □

Problem 7. Miki wants to distribute 75 identical candies to the students in her class such that each student gets at least 1 candy. For what number of students does Miki have the greatest number of possible ways to distribute the candies?

Solution. Suppose that we have the most number of ways to distribute the candies for n students.

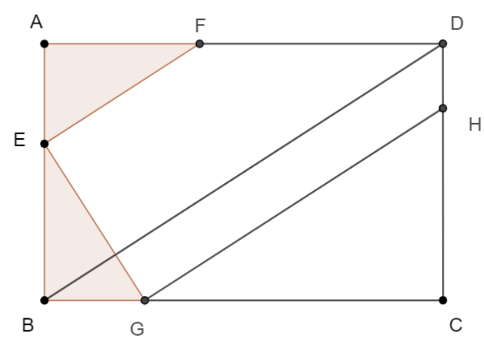
Then we start by giving each student a candy, so we have $75 - n$ candies left. The remaining problem is equivalent to the question of arranging $n - 1$ dividers and $75 - n$ candies in a row; the first student gets however many candies there are before the first divider (which may or may not be zero), the second student gets however many candies there are after the first divider and before the second; so on and so forth.

Therefore there are $(n - 1) + (75 - n)$ items in total, which makes 74 items. The number of ways to arrange them is $\binom{74}{n-1}$, which is clearly maximized when $n = \boxed{38}$.

For more information on this method of solving combinatorics problems, we recommend searching *stars and bars* online. □

Problem 8. Let $ABCD$ be a rectangle. Let points E, F, G, and H lie on the segments $\overline{AB}$, $\overline{AD}$, $\overline{BC}$, and $\overline{CD}$ (respectively) such that both $\overline{EF}$ and $\overline{GH}$ are parallel to $\overline{BD}$. If $\triangle AFE$ is congruent to $\triangle BEG$ and $\frac{AE}{HC} = \frac{1}{2}$, what is $\frac{AB}{BC}$?

Solution. Once again, we begin by drawing an accurate diagram:



Since we're only looking for ratios, without loss of generality assume that $AE = 1$ and $HC = 2$. Let the length of segment AF be x.

It follows that $EB = x$ and $BG = 1$ by the assumption that $\triangle AFE$ and $\triangle BEG$ are congruent. Further note that $\angle AFE = \angle ADB = \angle DBC = \angle CGH$, as lines $\overline{EF}, \overline{BD}$, and $\overline{GH}$ are parallel and as $ABCD$ is a rectangle. It follows that $\angle GCH = \angle FAE = 90°$ and $\angle AFE = \angle CGH$, so triangle HCG and EAF are similar with scale factor 2, as $HC = 2 \cdot AE$.

From this triangle similarity we see that $GC = 2x$, as $AF = x$. Observe that as $\triangle BCD$ and $\triangle FAE$ are similar, $\frac{DC}{CB} = \frac{AE}{AF}$. But $DC = AB = AE + EB = 1 + x$, $BC = BG + GC = 1 + 2x$, $AE = 1$, and $AF = x$, so

$$\frac{1+x}{1+2x} = \frac{1}{x}.$$

Cross-multiplication yields $1 + 2x = x + x^2$, or equivalently, $x^2 - x + 1 = 0$. The quadratic formula yields the solution $x = \frac{1+\sqrt{5}}{2}$.

Finally, we know that $\frac{AB}{BC} = \frac{1}{x}$, so our answer is $\frac{2}{1+\sqrt{5}} = \frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = \boxed{\frac{\sqrt{5}-1}{2}}$. □

Problem 9. If $a_1, a_2, a_3, \ldots$ is a geometric sequence satisfying

$$\frac{19a_{2019} + 19a_{2021}}{a_{2020}} = 25 + \frac{6a_{2006} + 6a_{2010}}{a_{2008}}$$

and $0 < a_1 < a_2$, what is the value of $\frac{a_2}{a_1}$?

Solution. Let k be the common ratio of the geometric sequence. We know that $a_i = a_1 k^{i-1}$ for all i, and making these substitutions into the equation yields

$$\frac{19a_1k^{2018} + 19a_1k^{2020}}{a_1k^{2019}} = 25 + \frac{6a_1k^{2005} + 6a_1k^{2009}}{a_1k^{2007}}.$$

This allows us to do many cancellations. Firstly, all the a_1 terms drop out, and secondly, most of the k-powers drop out, too. The equation becomes

$$19\left(\frac{1}{k} + k\right) = 25 + 6\left(\frac{1}{k^2} + k^2\right).$$

We make a substitution, as this is still hard to deal with: motivated by the fact that $(k+\frac{1}{k})^2 = k^2 + \frac{1}{k^2} + 2$, we let $k + \frac{1}{k} = x$. Our equation becomes

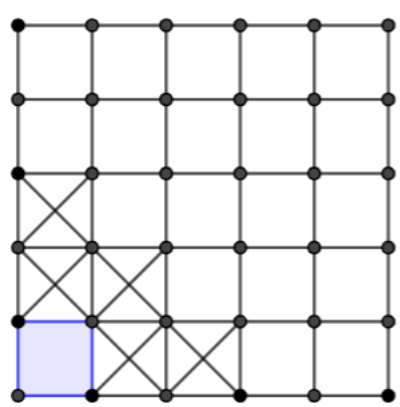
$$19x = 25 + 6(x^2 - 2).$$

This is much easier to deal with. The quadratic becomes $6x^2 - 19x + 13 = 0$, and it becomes obvious that one of the roots is 1. If $x = 1$, then $k + \frac{1}{k} = 1$ and $k < 1$, which contradicts $0 < a_1 < a_2$ as $a_2 = a_1k$. Therefore we find the other root, which is $x = \frac{13}{6}$.

Now we see that $k + \frac{1}{k} = \frac{13}{6}$, which yields roots $k = \frac{2}{3}$ and $k = \frac{3}{2}$. Again, if $k = \frac{2}{3}$, then $a_1 > a_2$; therefore $k = \frac{3}{2}$. But we observe that $\frac{a_2}{a_1} = k$, so our answer is $\boxed{\frac{3}{2}}$. □

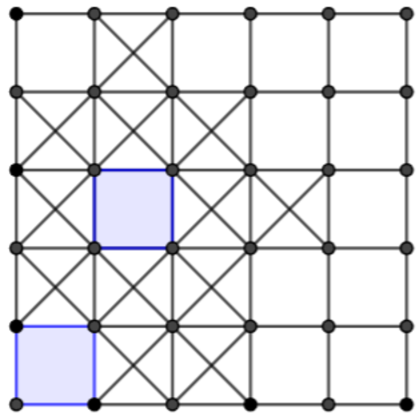
Problem 10. Elizabeth has an infinite grid of squares. (Each square is next to the four squares directly above it, below it, to its left, and to its right.) She colors in some of the squares such that the following two conditions are met: (1) no two colored squares are next to each other; (2) each uncolored square is next to exactly one colored square. In a 20×20 subgrid of this infinite grid, how many colored squares are there?

Solution. In problems like these, it's always a good idea to start with small cases. Here's a diagram where we've colored in exactly one square — the bottom left one:

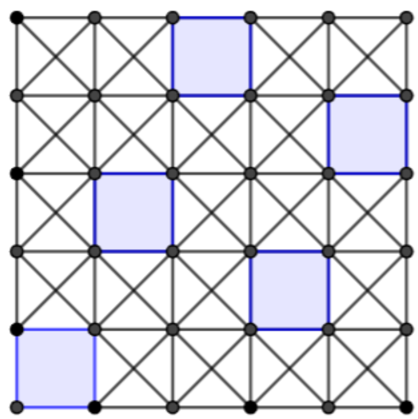


Observe that after coloring that square, all the squares marked with Xs become off limits. For the two squares directly next to the blue square, it's obvious that they can't be colored because of condition (1). As for the three other squares, if any of them are colored, then there exist uncolored squares directly next to two colored squares, contradicting condition (2).

Now, we see that to address condition (2) for the X-marked square two to the right and two up, we need a colored square directly above or to the right of it. Assume we choose above, without loss of generality. Our diagram looks like this:



Observe that to satisfy condition (2) for the X-marked square three to the right and two up from the bottom corner, we actually need a blue square to the right of it. We can fill in the rest of this five-by-five subgrid in a similar manner, and in the end it looks like this:



Observe that when we color in the grid this way, with our original blue square in the bottom left, that its closest neighbors on the same row or column must be at least five spaces away. Indeed, given the arguments we've made so far, it is impossible for any of the other squares in the bottom row or leftmost column to be shaded.

Moreover, the squares exactly five away from that blue square, left or up, must be shaded; otherwise the bottom-right corner and top-left corner squares will not have a colored neighbor.

Now the key observation is that we can just stack these five-by-five subgrids on top of one another to complete a 20-by-20 grid, and that to get a 20-by-20 grid, we need 16 subgrids total. Therefore the total number of colored squares is $5 \cdot 16 = \boxed{80}$. □

Problem 11. Find the smallest whole number $N \geq 2020$ such that N has twice as many even divisors as odd divisors and N^2 has a remainder of 1 when it is divided by 15.

Solution. Firstly, we claim that N has twice as many even divisors as odd divisors if $N = 4k$ for some odd number k. This is true because if a is an odd divisor of $4k$, then $2a$ and $4a$ must also be divisors of $4k$; it follows that for each odd divisor there must be two even divisors.

Next, $N^2 = (4k)^2 = 16k^2$. But $16k^2 \equiv k^2 \pmod{15}$, so we need $k^2 \equiv 1 \pmod{15}$, which for odd k occurs only when $k \equiv 1 \pmod{15}$ or $k \equiv 11 \pmod{15}$.

If $k \equiv 1 \pmod{15}$ and $k \equiv 0 \pmod{4}$, then $k \equiv 16 \pmod{60}$; otherwise, if $k \equiv 11 \pmod{15}$ and $k \equiv 0 \pmod{4}$, $k \equiv 56 \pmod{60}$.

The greatest multiple of 60 less than 2020 is 1980. Therefore we first check $1980 + 56 = 2036$. We know already that it satisfies the condition that $2036^2 \equiv 1 \pmod{15}$; we just need to check that 2036 has twice as many even divisors as odd ones. Indeed, $2036 = 4 \cdot 509$, and 509 is odd, so our answer is $\boxed{2036}$. □

Problem 12. We say that the sets A, B, and C form a "sunflower" if $A \cap B = A \cap C = B \cap C$. ($A \cap B$ denotes the intersection of the sets A and B.) If A, B, and C are independently randomly chosen 4-element subsets of the set $\{1, 2, 3, 4, 5, 6\}$, what is the probability that A, B, and C form a sunflower?

Solution. Firstly, the total number of ways to choose three 4-element subsets is $\binom{6}{4}^3 = 15^3$.

Next, we check cases on how large the common intersection $A \cap B \cap C$ must be. If $|A \cap B \cap C| = 1$, then there are five elements left over, and none can be shared between any two of A, B, and C; this clearly does not work. Next, if $|A \cap B \cap C| = 2$, then we run into the same problem: there are two elements left in A, B, and C each, but there are four elements left of $\{1, 2, 3, 4, 5, 6\}$.

Therefore assume firstly that $|A \cap B \cap C| = 3$. There are $\binom{6}{3} = 20$ ways to choose this common intersection, and then each of the three remaining elements must be assigned to one of A, B, or C. This yields $6 \cdot 20$ total arrangements that form a sunflower.

There is one more case: where all the sets are the same. This yields $\binom{6}{4} = 15$ more arrangements that form a sunflower.

Our final answer is thus $\frac{15+120}{15^3} = \frac{9}{15^2} = \boxed{\frac{1}{25}}$. □