Team Round: Graph Theory

Girls in Math at Yale

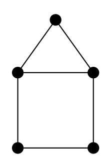
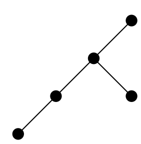
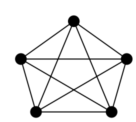
February 1, 2020

The objective of this team round is to give you a taste of graph theory, a major branch of discrete mathematics. The first section introduces the concept of a graph and some basic properties. The second section concerns special graphs called trees and bipartite graphs. The third section explores ways of "coloring" graphs. It is possible to do most of the third section before the second section, if you so choose.

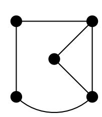
Note that questions are generally worth more points as the test progresses. There are 100 points in total. In your solutions, you may refer to previous problems, even if you have not solved them. Questions that ask you to "show" should have explanations (proofs); otherwise, answers without explanations are sufficient. Please submit each solution on a separate sheet of paper. Have fun!

1 Warm-Up: Graphs and Degrees (30 points)

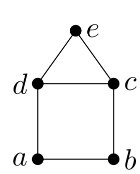
For the purpose of this team round, a *graph* consists of a collection of *vertices* (dots) and some *edges* (lines) connecting pairs of vertices. (This notion of a graph is completely unrelated to the graph of a function.) Here are three drawings of different graphs, each with 5 vertices:

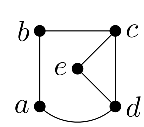




These graphs have (from left to right) 6, 4, and 10 edges. The exact position of the vertices does not matter; all that matters is which vertices are connected to which other vertices. So, for example, we can also draw the graph on the left (which we will call the "house" graph) like this:



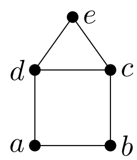
More formally, we say that a graph is a set of vertices, along with the set of pairs of vertices that are connected by edges. In the previous example, we can label the vertices with the letters a, b, c, d, e:

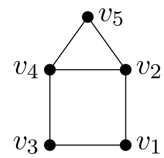




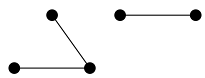
The following pairs of vertices are connected by edges in each drawing: $\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}$, $\{c, e\}, \{d, e\}$. We write a graph (called G) like this: $G = (V, E)$, where V is the set of vertices of the graph and E is the set of its edges (i.e., the pairs of vertices that are connected). In the example above, we have $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{c, e\}, \{d, e\}\}$.

Usually, we don't bother giving names to the vertices. In this case, we don't distinguish between different ways of labelling the vertices, so we consider the following graphs to be the same:





We say that a graph G is *connected* if every pair of vertices in G is connected by a path along the edges of G. All of the example graphs above are connected. Here is a graph (on 5 vertices) that is not connected:



Graphs have a wide variety of applications. For instance, you can make a graph where the vertices correspond to the people in a room, and two vertices are connected by an edge if the people know each other. Without further ado, a few problems:

Problem 1.1 (6 points)**.** *Draw all* 11 *of the (unlabeled) graphs with* 4 *vertices. (There are* 6 *connected graphs and* 5 *non-connected graphs.)*

Problem 1.2 (2 points)**.** *What is the maximum number of edges in a graph with n vertices?*

Problem 1.3 (2 points)**.** *What is the minimum number of edges in a graph with n vertices where every vertex is touching at least* 1 *edge? (Your answer should depend on whether n is even or odd.)*

We define the *degree* of a vertex to be the number of edges that it is touching. We write $d(v)$ to denote the degree of the vertex v. In the "house" graph above, for instance, we have $d(v_1) = 2, d(v_2) = 3, d(v_3) = 2, d(v_4) = 3, d(v_5) = 2$. (If the vertex v is not touching any edges, then $d(v) = 0$.) The vertex degrees tell us a lot about the structure of the graph!

Problem 1.4 (6 points)**.** *Let $G = (V, E)$ be a graph with r edges. Find an expression for*

$$\sum_{v \in V} d(v)$$

(the sum of the degrees of the vertices of G) in terms of r, and briefly show why this is the case. (This problem is important and will help you in the rest of the team round; if you can't figure it out, come "buy" the formula for 2 points, and then you can think about why it works.)

Problem 1.5 (4 points)**.** *Show that every graph has an even number of vertices with odd degree.*

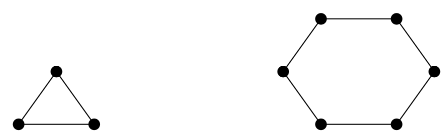
Problem 1.6 (10 points)**.** *Is it possible to have a graph whose vertices have the following degrees? For each part, either draw a graph with these degrees or explain why no such graph exists.*

- $1, 1, 1, 2, 3, 3, 4$ *(7 vertices)*

- $1, 1, 1, 2, 5$ *(5 vertices)*
- $1, 1, 1, 1, 3, 3, 4, 4$ *(8 vertices)*
- $3, 3, 5, 5, 5, 5$ *(6 vertices)*
- $1, 1, 1, 1, 2$ *(5 vertices)*

2 Trees and Bipartite Graphs (30 points)

In this section, we will meet two common and important types of graphs. Given a graph G, we can make a path between two vertices by travelling along edges of the graph. Sometimes, we can make a path that ends in the same place where it started, and this is called a *cycle*. We define the *length* of a cycle to be the number of edges in it, and we write C_k for the cycle of length k. (This exists for every integer $k \geq 3$.) For example, here are drawings of C_3 and C_6:



For example, the "house" graph has a cycle of length 3 (going through the vertices v_2, v_4, v_5), a cycle of length 4 (going through the vertices v_1, v_2, v_4, v_3), and a cycle length 5 (going through the vertices v_1, v_2, v_5, v_4, v_3), and these are the only cycles.

Now, we say that a graph is a *tree* if it is connected and it has no cycles at all. Among the three graphs on page 1, only the middle one is a tree. Here are two more examples of trees:

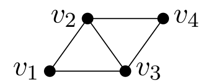
Problem 2.1 (4 points)**.** *Draw all* 6 *of the trees on* 6 *vertices.*

Problem 2.2 (2 points)**.** *How many edges are there in a tree with n vertices?*

Problem 2.3 (4 points)**.** *Show that every tree with at least* 2 *vertices has at least* 2 *vertices of degree exactly* 1*. (These vertices are called* leaves*.)*

We now move from trees to bipartite graphs. A graph $G = (V, E)$ is called *bipartite* if it is possible to divide the vertices V into two groups A and B such that every edge has one end vertex in A and the other end vertex in B. Here are some examples of bipartite graphs (with the vertices of A drawn on the left and the vertices of B drawn on the right):

Note that sometimes there can be multiple possible ways to divide the vertices into A and B; this is completely okay. Here is an example of a graph that is not bipartite:



Here is why this graph is not bipartite: suppose v_1 is in A; since there are edges connecting v_1 to v_2 and v_3, we must put both v_2 and v_3 in B, but then the edge between v_2 and v_3 shows that this division of the vertices does not work; thus, there is *no* way to divide the vertices into groups A and B in such a way that every edge has one endpoint in A and the other endpoint in B.

Problem 2.4 (6 points)**.** *Show that all trees are bipartite.*

Problem 2.5 (4 points)**.** *What is the maximum number of edges in a bipartite graph with n vertices? (Your answer should depend on whether n is even or odd.)*

Problem 2.6 (10 points)**.** *Show that a graph G is bipartite if and only if there are no cycles of odd length. You may consider only connected graphs (Hint: think about erasing edges of G until you are left with a tree, which should give you ideas about how to divide the vertices of G into groups A and B.)*

3 Coloring Graphs (40 points)

One thing that mathematicians like to do with graphs is color the vertices. It is easier to use the colors $1, 2, 3, \ldots$ instead of the actual colors red, green, blue, etc. Such a labeling of the vertices of a graph is called a *coloring*. We say that a coloring of a graph G is *proper* if every edge has endpoints of different colors. Here is an example of a proper coloring (on the left) and a non-proper coloring (on the right), each using the "colors" $1, 2, 3$:

(The issue with the coloring on the right is that the vertical edge on the right has both endpoints labeled 2.)

Here is a nice question: given a graph G, what is the minimum number of colors needed in order to make a proper coloring of the vertices? We define the *chromatic number* of G (denoted $\chi(G)$) to be the minimum number of colors with which it is possible to make a proper coloring. In order to show that a particular graph G has chromatic number k, you have to do two things:

(a) Make a proper coloring (just write it down!) using the colors $1, 2, \ldots, k$. This shows that $\chi(G) \leq k$.

(b) Show that every possible coloring with the colors $1, 2, \ldots, k-1$ is non-proper. This shows that $\chi(G) > k - 1$.

The "house" graph above has chromatic number 3, which we can check as follows:

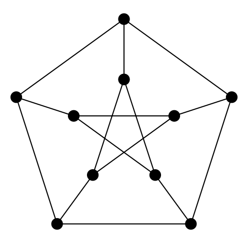
(a) Above, we exhibited a proper coloring with the colors $1, 2, 3$.

(b) Consider any coloring using only the colors $1, 2$. Think about the 3 vertices that make up the triangle at the "top" of the graph. These vertices cannot all be different colors (since there are 3 vertices and only 2 colors), so there is an edge between these 2 vertices of the same color. But then it is impossible for such a coloring to be proper.

Here are some examples of chromatic numbers to think about:

Problem 3.1 (18 points)**.** *Compute the chromatic number of each of the following graphs (no explanation necessary):*

- *a cycle of even length*
- *a cycle of odd length*
- *the graph with n vertices and no edges*
- *the graph with n vertices, where there is an edge connecting every pair of vertices (called the* complete graph *on n vertices)*
- *this graph (called the "Petersen graph"):*



- *any bipartite graph with at least 1 edge*

Problem 3.2 (4 points)**.** *Show that if a graph G contains ω vertices where every pair is connected by an edge, then G satisfies $\chi(G) \geq \omega$. (The number ω is called the* clique number *of the graph G.)*

Problem 3.3 (8 points)**.** *Show that if G is a graph where every vertex has degree less than or equal to D, then $\chi(G) \leq D + 1$. (Hint: think of an algorithm for making a proper coloring using the colors $1, \ldots, D + 1$; try assigning colors to the vertices one-by-one.)*

Problem 3.4 (10 points)**.** *Show that if G is a graph with exactly r edges, then the chromatic number of G satisfies*

$$\frac{\chi(G)(\chi(G) - 1)}{2} \leq r.$$

(Hint: choose a proper coloring with $1, 2, \ldots, \chi(G)$, then look at the groups of vertices with each color and think about what it means for all of these colors to be necessary.)