Team Round: Permutations

Girls in Math at Yale

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The objective of this team round is to teach you about permutations, which are an important concept in many branches of mathematics. The first section introduces the basic concepts of permutations. The second section explores pattern avoidance, an interesting topic which is the subject of a lot of current mathematical research. The third section presents the very useful "cycle notation" for permutations and explores some problems related to a special type of permutation called a transposition. The second and third sections are completely independent and can be done in either order.

Note that the questions are generally worth more points as the test progresses. There are 80 points total. In your solutions, you may refer to previous problems, even if you have not solved them. Questions that ask you to "show" should have explanations (proofs); otherwise, answers without explanations are sufficient. Please submit each solution on a separate sheet of paper. Have fun!

1 What Are Permutations? (12 points)

A permutation is a way to write the integers 1 through n in some order. For example, 123, 312, and 426351 are permutations, with $n = 3$, $n = 3$, and $n = 6$, respectively. This way of writing permutations is called *word notation*. (Later, we will introduce another notation for permutations–for now, don't worry about this.) We will call a permutation of the integers 1 through n a *permutation on n elements*. There are $n!$ permutations on n elements: there are n ways to choose the first number in the word notation, then $n - 1$ ways to choose the next number, and so on, all the way to just 1 way to choose the last number.

Problem 1.1 (2 points)**.** *Write down all of the permutations on 4 elements in word notation.*

At this point, you have all the background you need for Section 2. The rest of this section is devoted to thinking of permutations as functions, which will be essential for sections 3 and 4.

We can also think of the integers 1 through n as a function p from the set $\{1, 2, \ldots, n\}$ to itself such that for each element y in $\{1, 2, \ldots, n\}$, there exists a unique x in $\{1, 2, \ldots, n\}$ with $p(x) = y$. Equivalently, $\{p(1), p(2), \ldots, p(n)\} = \{1, 2, \ldots, n\}$ as sets.

For example, when $n = 4$, one possible permutation of the integers 1 through 4 is the function p defined by $p(1) = 3$, $p(2) = 2$, $p(3) = 4$, and $p(4) = 1$. We can summarize the action of this permutation p in the following *two-row diagram*:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

In general, a permutation on n elements has the following two-row diagram:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \downarrow & \downarrow & \cdots & \downarrow \\ p(1) & p(2) & \cdots & p(n) \end{pmatrix}.$$

Note that only the bottom row actually contains information about the function p. With this in mind, we can record all of the information about p by writing down the bottom row (as a string of consecutive numbers). This is just a permutation in word notation as defined in the first paragraph of this section! More precisely, the word notation of p is the string $p(1)p(2)\cdots p(n)$. In the example permutation above, the word notation is 3241.

The word notation of a permutation is unique, and any rearrangement of the numbers 1 through n is the word notation of a unique permutation on n elements. Note that in the word notation of a permutation on n elements, the integers 1 through n all appear exactly once. So you can think of a permutation as a way to write the numbers 1 through n in some order.

Other examples of permutations in word notation are 231 (the permutation on 3 elements with $p(1) = 2$, $p(2) = 3$, and $p(3) = 1$) and 12345 (the permutation on 5 elements where $p(x) = x$ for all $x \in \{1, 2, 3, 4, 5\}$).

Problem 1.2 (2 points). *How many permutations on* 7 *elements send the numbers* 1, 2, *and* 3 *to* 5, 6, *and* 7 *(in some order)?*

As we mentioned before, permutations are functions. Like many other functions, they can be applied one after the other. If p and q are both permutations on n elements, then the *composition* $p \circ q$ is a function from $\{1, 2, \dots n\}$ to itself defined by $(p \circ q)(x) = p(q(x))$ for all $x \in \{1, 2, \dots n\}$. That is, you apply q and then you apply p; note that you apply the permutation on the RIGHT first. We will also call the composition of permutations the *product*.

For example, let $p = 3241$ and $q = 2143$ be permutations on 4 elements. Then $(p \circ q)(1) = p(q(1)) = p(2) = 2$. Similarly, $(p \circ q)(2) = p(q(2)) = p(1) = 3$, $(p \circ q)(3) = p(q(3)) = p(4) = 1$, and $(p \circ q)(4) = p(q(4)) = p(3) = 4$. So in word notation, we end up with $p \circ q = 2314$.

(It's worth taking a moment to explain why the composition of two permutations on n elements is another permutation on n elements. Recall the first definition we gave of a permutation on n elements: a function p such that for each element $y \in \{1, 2, \dots, n\}$, there exists a unique $x \in \{1, 2, \dots, n\}$ with $p(x) = y$. Let p and q be permutations on n elements, and we will check that

$p \circ q$ satisfies this definition. Choose any $y \in \{1, 2, \ldots, n\}$. Since p is a permutation, there exists a unique element $z \in \{1, 2, \ldots n\}$ such that $p(z) = y$. Then $p(q(w)) = y$ if and only if $q(w) = z$. But since q is a permutation, there exists a unique $x \in \{1, 2, \ldots, n\}$ such that $q(x) = z$. Putting this all together, we have shown that there is a unique element $x \in \{1, 2, \ldots, n\}$ such that $p(q(x)) = y$, so $p \circ q$ is a permutation on n elements.)

Note that it simply doesn't make sense to talk about the composition of permutations on different numbers of elements.

Problem 1.3 (4 points). *Find* $321 \circ 132$ *and* $42351 \circ 13524$. *(Write the resulting permutations in word notation.)*

Order matters in compositions of permutations, so in general you need to be careful about which permutation comes first.

Problem 1.4 (4 points). *Give an example of two permutations p and q on 3 elements where $p \circ q$ and $q \circ p$ are different.*

2 Pattern Avoidance (38 points)

In this section, the big idea is that we can think of a permutation on n elements as determining a relative order, or *pattern*, on a string of n numbers.

Permutations naturally *contain* copies of some permutations on fewer elements. For example, consider the permutation 23541 on 5 elements. If we pick out the first, third, and fifth numbers, we get 251. These have the same relative order as 231, so we can think of this as a copy of the permutation 231 (on 3 elements). Similarly, if we pick out the first, second, and third elements, then we get 235, which has the same relative order as 123.

Formally, we say that the permutation $p = a_1 a_2 \cdots a_n$ *contains* the pattern $q = b_1 b_2 \cdots b_m$ (where $m \leq n$) if we can find m of the a_i's (keeping them in order) such that their relative order is the same as $b_1 b_2 \cdots b_m$. Note that the a_i's we choose do NOT have to be consecutive.

To continue the example from above, we see that 23541 also contains the patterns 12 (from the first and second elements, for instance), 321 (from the third, fourth, and fifth elements), 2341 (from the first, second, fourth, and fifth elements), and 23541 (from taking all of the elements). (It contains more patterns, too; these are just examples).

We say that the permutation p *avoids* the pattern q (where p is a permutation on at least as many elements as q) if p does NOT contain q. In our main example, we can see that 23541 avoids the pattern 1234 because we cannot find 4 increasing numbers in the string 23541.

Problem 2.1 (4 points). *Which permutations (patterns) on 3 elements are contained by 13254?*

Problem 2.2 (4 points). *Find all permutations on 4 elements that avoid the pattern 321.*

Problem 2.3 (6 points)**.** *Find a permutation on* 5 *elements that contains all of the patterns of length* 3*.*

We can also talk about permutations that avoid multiple patterns simultaneously (that is, permutations that don't contain any of a number of patterns).

Problem 2.4 (6 points)**.** *Find* 3 *different permutations on* 8 *elements that avoid both the pattern* 123 *and the pattern* 132*.*

Problem 2.5 (6 points)**.** *Show that every permutation on* 6 *elements contains either the pattern* 123 *or the pattern* 321*.*

In modern research, people often study how many permutations of a specified length contain or avoid certain patterns. The following two problems give a taste of these kinds of ideas.

Problem 2.6 (6 points)**.** *For $n \geq 5$, let $f(n)$ be the number of permutations on n elements that contain the pattern* 13524*, and let $g(n)$ be the number of permutations on n elements that contain the pattern* 53142*. Show that $f(n) = g(n)$ for all integers $n \geq 5$.*

Problem 2.7 (6 points)**.** *For $n \geq 4$, let $f(n)$ be the number of permutations on n elements that avoid the pattern* 123*, and let $g(n)$ be the number of permutations on n elements that avoid the patterns* 1234*,* 1243*,* 1324*,* 1342*,* 1423*,* 2134*,* 2314*,* 2341*,* 3124*, and* 4123*. Show that $f(n) = g(n)$ for all integers $n \geq 4$.*

3 Cycles and Transpositions (30 points)

Word notation is nice, but it's often useful to express permutations differently. First, we consider the example $p = 245136$. Note that $p(1) = 2$, $p(2) = 4$, and $p(4) = 1$ brings us back to where we started. Then we call $1 \to 2 \to 4 \to 1$ a *cycle* of p of length 3. Similarly, $p(3) = 5$ and $p(5) = 3$, so $3 \to 5 \to 3$ is another cycle of p, this time of length 2. Finally, $p(6) = 6$ can be seen as a cycle of length 1. (As will be explained below, these three cycles tell us all of the information about the function p because we can use them to determine $p(x)$ for all $x \in \{1, 2, 3, 4, 5, 6\}$.) In order to record all cycles of length greater than 1 in a compact notation, we can write (124)(35). (The cycles of length 1–in this case, just (6)– are implicit.)

This idea forms the basis of cycle notation, which is obtained for a permutation p on n elements as follows. First, consider $p(1)$, then apply p to that, and so on, until you get back to 1. Write down these elements in parentheses as the first cycle. (In the example of the previous paragraph, this was (124).) Next, take the smallest integer (if there is one) that was not used in the first cycle, and use it as the starting point for the next cycle. (In the example above, this was (35).) Continue this process until you have used up all of the integers from 1 to n. Finally, delete the cycles of length 1, and what's left is the cycle permutation of the permutation.

As another example, the cycle notation of 13247658 is (23)(57).

Cycle notation can easily be converted back into word notation. For example, let (152)(46) be the cycle notation for a permutation p on 7 elements. This means that $1 \to 5 \to 2 \to 1$ and $4 \to 6 \to 4$. Since 3 and 7 do not appear in the cycle notation, they must be in cycles of length 1. Now we can easily read off $p(1) = 5$, $p(2) = 1$, $p(3) = 3$, $p(4) = 6$, $p(5) = 2$, $p(6) = 4$, and $p(7) = 7$, so $p = 5136247$. Cycle notation is also unique, so word notation and cycle notation are completely equivalent ways to express permutations.

A few important notes:

1. Within each cycle, the smallest element is always listed first. And the first elements of the cycles increase from left to right.

2. In cycle notation, every number from 1 to n appears either once (if it is part of a cycle of length at least 2) or not at all (if it forms a cycle of length 1).

3. When using cycle notation, it is important to specify how many elements the permutation is on. For example, the cycle notation of 21 and 213 are both just (12).

4. What about the permutation $123 \ldots n$ that sends every element to itself? Here all of the cycles are of length 1, so, according to the description above, they should all be deleted, and the cycle notation should be blank. For convenience of notation, this special permutation (called the *identity* permutation) is written (1) in cycle notation.

Problem 3.1 (6 points)**.** *Write* 3512647 *in cycle notation. Write* (13)(24) *(as a permutation on* 4 *elements) in word notation. Write* (245)(68) *(as a permutation on* 9 *elements) in word notation.*

Problem 3.2 (4 points)**.** *Find* $(13) \circ (23)$ *and* $(142)(35) \circ (254)$*. (Give your answers in cycle notation.)*

The following property of the product of *disjoint* cycles is very useful.

Problem 3.3 (6 points)**.** *Show that* $(a_1 a_2 \cdots a_k) \circ (b_1 b_2 \cdots b_\ell) = (b_1 b_2 \cdots b_\ell) \circ (a_1 a_2 \cdots a_k)$ *if* $a_i \neq b_j$ *for all* $1 \leq i \leq k$ *and* $1 \leq j \leq \ell$*. (Hint: think about word notation. Where is* a_i *sent? And* b_j*? What about elements that are neither an* a_i *nor a* b_j*?)*

A *transposition* is a permutation whose cycle notation is a single cycle of length 2. In other words, a transposition is a permutation with cycle notation (ab) for some distinct $a, b \in \{1, 2, \ldots, n\}$. The transposition (ab) is easy to work with because it swaps a and b and doesn't affect the other elements. The purpose of this section is to show that, in a sense, transpositions are the basic "building blocks" of all permutations.

Problem 3.4 (2 points)**.** *Write* (1324) *as products of transpositions.*

Problem 3.5 (4 points). *Write the cycle* $(a_1a_2\cdots a_k)$ *as the product of transpositions. (Hint: you need only* $k-1$ *transpositions.)*

Problem 3.6 (4 points). *Show that every permutation can be written as a product of transpositions.*

Problem 3.7 (4 points). *Show that every permutation on* n *elements can be written as a product of transpositions of the form* $(1a)$*, where* a *ranges from* 2 *to* n*. (Hint: first try to write the transposition* (bc) *in this way.)*