Mathathon Round 1 (2 points each)

1. A small pizza costs \$4 and has 6 slices. A large pizza costs \$9 and has 14 slices. If the MMATHS organizers got at least 400 slices of pizza (having extra is okay) as cheaply as possible, how many large pizzas did they buy?

2. Rachel flips a fair coin until she gets a tails. What is the probability that she gets an even number of heads before the tails?

3. Find the unique positive integer n that satisfies $n! \cdot (n+1)! = (n+4)!$.

Mathathon Round 2 (3 points each)

4. The Portland Malt Shoppe stocks 10 ice cream flavors and 8 mix-ins. A milkshake consists of exactly 1 flavor of ice cream and between 1 and 3 mix-ins. (Mix-ins can be repeated, the number of each mix-in matters, and the order of the mix-ins doesn't matter.) How many different milkshakes can be ordered?

5. Find the minimum possible value of the expression $(x)^2 + (x+3)^4 + (x+4)^4 + (x+7)^2$, where x is a real number.

6. Ralph has a cylinder with height 15 and volume $\frac{960}{\pi}$. What is the longest distance (staying on the surface) between two points of the cylinder?

Mathathon Round 3 (4 points each)

7. If there are exactly 3 pairs (x, y) satisfying $x^2 + y^2 = 8$ and $x + y = (x-y)^2 + a$, what is the value of a?

8. If n is an integer between 4 and 1000, what is the largest possible power of 2 that $n^4 - 13n^2 + 36$ could be divisible by? (Your answer should be this power of 2, not just the exponent.)

9. Find the sum of all positive integers $n \geq 2$ for which the following statement is true: "for any arrangement of n points in three-dimensional space where the points are not all collinear, you can always find one of the points such that the $n-1$ rays from this point through the other points are all distinct."

Mathathon Round 4 (6 points each)

10. Donald writes the number 12121213131415 on a piece of paper. How many ways can he rearrange these fourteen digits to make another number where the digit in every place value is different from what was there before?

11. A question on Joe's math test asked him to compute $\frac{a}{b} + \frac{3}{4}$, where a and b were both integers. Because he didn't know how to add fractions, he submitted $\frac{a+3}{b+4}$ as his answer. But it turns out that he was right for these particular values of a and b! What is the largest possible value that a could have been?

12. Christopher has a globe with radius r inches. He puts his finger on a point on the equator. He moves his finger 5π inches North, then π inches East, then 5π inches South, then 2π inches West. If he ended where he started, what is the largest possible value of r?

Mathathon Round 5 (7 points each)

13. Suppose $\triangle ABC$ is an isosceles triangle with $\overline{AB} = \overline{BC}$, and X is a point in the interior of $\triangle ABC$. If $m\angle ABC = 94^\circ$, $m\angle ABX = 17^\circ$, and $m\angle BAX = 13^\circ$, then what is $m\angle BXC$ (in degrees)?

14. Find the remainder when $\sum_{n=1}^{2019} 1 + 2n + 4n^2 + 8n^3$ is divided by 2019.

15. How many ways can you assign the integers 1 through 10 to the variables a, b, c, d, e, f, g, h, i, and j in some order such that $a < b < c < d < e$, $f < g < h < i$, $a < g$, $b < h$, $c < i$, $f < b$, $g < c$, and $h < d$?

Mathathon Round 6 (8 points each)

16. Call an integer n *equi-powerful* if n and n^2 leave the same remainder when divided by 1320. How many integers between 1 and 1320 (inclusive) are equi-powerful?

17. There exists a unique positive integer $j \leq 10$ and unique positive integers $n_j, n_{j+1}, \ldots, n_{10}$ such that

$$j \leq n_j < n_{j+1} < \cdots < n_{10}$$

and

$$\binom{n_{10}}{10} + \binom{n_9}{9} + \cdots + \binom{n_j}{j} = 2019.$$

Find $n_j + n_{j+1} + \ldots + n_{10}$.

18. If n is a randomly chosen integer between 1 and 390 (inclusive), what is the probability that $26n$ has more positive factors than $6n$?

Mathathon Round 7 (9 points each)

19. Suppose S is an n-element subset of $\{1, 2, 3, ..., 2019\}$. What is the largest possible value of n such that for every $2 \leq k \leq n$, k divides exactly $n-1$ of the elements of S?

20. For each positive integer n, let $f(n)$ be the fewest number of terms needed to write n as a sum of factorials. For example, $f(28) = 3$ because $4! + 2! + 2! = 28$ and 28 cannot be written as the sum of fewer than 3 factorials. Evaluate $f(1) + f(2) + \cdots + f(720)$.

21. Evaluate $\sum_{n=1}^{\infty} \frac{\phi(n)}{101^n - 1}$, where $\phi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n.