Math Majors of America Tournament for High Schools
2019 Individual Test Solutions

1. When Charles traveled from Hawaii to Chicago, he moved his watch 5 hours backwards instead of 5 hours forwards. He plans to wake up at 7:00 the next morning (Chicago time). When he wakes up during the night and sees that his watch says 6:00, how many more hours should he sleep? (He has a 12-hour watch, not a 24-hour watch.)

Answer: 3

Solution: The watch is 10 hours behind, so 6:00 PM on the watch is actually 4:00 AM Chicago time.

2. Rover's dog house in the middle of a large grassy yard is a regular hexagon with side length 10. His leash, which has length 20, connects him to one vertex on the outside of the dog house. His leash cannot pass through the interior of the dog house. What is the total area of the yard (i.e., outside the doghouse) that he can roam? (Give your answer in units squared.)

Answer: 300π

Solution: Draw a picture for different scenarios based on how many sides of the dog house his leash is flush with (i.e., "wraps around"). If the leash is flush with one side of the dog house, this gives an area of $(2)(\frac{1}{6})(100\pi) = \frac{100\pi}{3}$. Otherwise, the remaining area is $(\frac{2}{3})(400\pi) = \frac{800\pi}{3}$. In total, this is 300π.

3. Daniel rolls three fair six-sided dice. Given that the sum of the three numbers he rolled was 6, what is the probability that all of the dice showed different numbers?

Answer: $\frac{3}{5}$

Solution: There are $3! = 6$ ways to get a sum of 6 with all distinct summands (permutations of $1+2+3$). There is 1 way to get a sum of 6 with all summands the same $(2+2+2)$. There are 3 ways to get a sum of 6 with exactly 2 distinct summands (permutations of $1+1+4$). So the answer is $\frac{6}{6+1+3} = \frac{3}{5}$.

4. The points A, B, and C lie on a circle centered at the point O. Given that $m\angle AOB = 110^\circ$ and $m\angle CBO = 36^\circ$, there are two possible values of $m\angle CAO$. Give the (positive) difference of these two possibilities (in degrees).

Answer: 70°

Solution: Since $m\angle CBO = 36^\circ$, we know that $m\angle BOC = 180^\circ - 2(36^\circ) = 108^\circ$ and minor arc $\overset{\frown}{BC}$ subtends 108°. We now condition on whether or not C lies on minor arc $\overset{\frown}{AB}$. If so, then $m\angle AOC = 110^\circ - 108^\circ = 2^\circ$ and $m\angle CAO = \frac{1}{2}(180^\circ - 2^\circ) = 89^\circ$. Otherwise, $m\angle AOC = 360^\circ - 110^\circ - 108^\circ = 142^\circ$ and $m\angle CAO = \frac{1}{2}(180^\circ - 142^\circ) = 19^\circ$. So the desired quantity is $89^\circ - 19^\circ = 70^\circ$.

5. Joanne has four piles of sand, which weigh 1, 2, 3, and 4 pounds, respectively. She randomly chooses a pile and distributes its sand evenly among the other three piles. She then chooses one of the remaining piles and distributes its sand evenly among the other two. What is the expected weight (in pounds) of the larger of these two final piles?

Answer: $\frac{35}{6}$

Solution: The sand from the two piles that get split up end up contributing evenly to the final two piles. Thus, the expected difference in weight between the last two piles is the expected difference between two randomly chosen piles. This is $(\frac{1}{6})(3) + (\frac{2}{6})(2) + (\frac{3}{6})(1) = \frac{10}{6} = \frac{5}{3}$. The total amount of sand is $1+2+3+4 = 10$. So on average, the larger pile has weight $5 + (\frac{1}{2})(\frac{5}{3}) = 5 + \frac{5}{6} = \frac{35}{6}$.

6. When $15!$ is converted to base 8, it is expressed as $\overline{230167356abc00}$ for some digits a, b, and c. Find the missing string $\overline{abc}$.

Answer: 540

Solution: First, note that c is in the $8^3 = 2^9$ place. By counting 2's, it's easy to see that 2^{11} divides $15!$, so we must have $c = 0$. In base 8, the divisibility rule for 7 is that the sum of the digits is a multiple of 7. Since $15!$ is divisible by 7, we get that

$2+3+0+1+6+7+3+5+6+a+b+0+0+0 = 33+a+b$ is a multiple of 7, i.e., $a+b$ is either 2 or 9. Similarly, the base-8 divisibility rule for 9 is that the alternating sum of the digits is a multiple of 9, so $2-3+0-1+6-7+3-5+6-a+b-0+0-0 = 1-a+b$ is a multiple of 9, i.e., $b-a=-1$. Substituting $b=a-1$ into the first equation gives that $2a-1$ is either 2 or 9. Since $2a-1$ is odd, we conclude that $2a-1=9$, so $a=5$. Then $b=5-1=4$, and the answer is 540.

7. Construct triangles $\triangle ABC$ and $\triangle A'B'C'$ such that $\overline{AB}=10$, $\overline{BC}=11$, $\overline{AC}=12$, C lies on segment $\overline{A'A}$, B lies on $\overline{C'C}$, A lies on $\overline{B'B}$, and $\overline{A'C}=\overline{C'B}=\overline{B'A}=1$. Find the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$.

Answer: $\frac{343}{264}$

Solution: Let $\overline{AB}=c, \overline{BC}=a, \overline{AC}=b$, and $[ABC]=k$. Now draw line segments $\overline{A'B}$, $\overline{C'A}$, $\overline{B'C}$. Then by area ratios, we know that $[A'BC]=\frac{[ABC]}{b}=\frac{k}{b}$, and $[A'BC']=\frac{[A'BC]}{a}=\frac{k}{ab}$. Similarly, we can find that $[AB'C]=\frac{k}{c}$, $[A'B'C]=\frac{k}{bc}$, $[ABC']=\frac{k}{a}$, and $[AB'C']=\frac{k}{ac}$. Thus, $[A'B'C']=k+\frac{k}{a}+\frac{k}{b}+\frac{k}{c}+\frac{k}{ab}+\frac{k}{bc}+\frac{k}{ac}=k\frac{abc+ab+bc+ac+a+b+c}{abc}=k\frac{(a+1)(b+1)(c+1)-1}{abc}$, so the ratio of the areas is $\frac{(a+1)(b+1)(c+1)-1}{abc}$. Plugging in the values for a,b,c, we get that the answer is $\frac{343}{264}$.

8. Given that $x^4+y^4+z^4=1$, let a be the maximum possible value of $x+y+z$, let b be the minimum possible value of $x+y+z$, let c be the maximum possible value of $x-y-z$, and let d be the minimum possible value of $x-y-z$. What is the value of $abcd$?

Answer: 27

Solution: Clearly, all of our extremal solutions will occur when $|x|=|y|=|z|=(\frac{1}{3})^{\frac{1}{4}}$. (This can also be done many other ways, for instance, with Lagrange multipliers.) Then we can read off $a=c=3\cdot(\frac{1}{3})^{\frac{1}{4}}$ and $b=d=-3\cdot(\frac{1}{3})^{\frac{1}{4}}$. Then $abcd=81\cdot\frac{1}{3}=27$.

9. How many (possibly empty) subsets of $\{1,2,3,4,5,6,7,8,9,10,11\}$ do not contain any pair of elements with difference 2?

Answer: 273

Solution: Clearly, we can consider the odds and evens separately. Let a_n denote the number of subsets of $\{1,2,\ldots,n\}$ that do not contain two consecutive integers. Then the answer to our problem is a_5a_6. Clearly, $a_0=1$ (the empty set) and $a_1=2$. For $n\geq 2$, we condition on whether or not n is included: if n is not included, then there are a_{n-1} options, and if n is included, then $n-1$ is excluded and there are a_{n-2} options. So $a_n=a_{n-1}+a_{n-2}$. This lets us compute $a_2=3$, $a_3=5$, $a_4=8$, $a_5=13$, $a_6=21$. So $a_5a_6=13\cdot 21=273$.

10. The positive real numbers x and y satisfy $x^2=y^2+72$. If x^2, y^2, and $(x+y)^2$ are all integers, what is the largest possible value of x^2+y^2?

Answer: 650

Solution: Note that $x^2+y^2=(x^2-y^2)+2y^2=72+2y^2$, so we are trying to maximize the value of y^2. Let $z=(x+y)^2$, so that $z=(\sqrt{y^2+72}+y)^2=2y^2+72+2\sqrt{y^4+72y^2}$. In particular, $y^4+72y^2=n^2$ must be a perfect square. Factoring gives $(y^2+36+n)(y^2+36-n)=1296$. Recall that y^2 is maximized when n is maximized, i.e., the difference between the two factor is maximized. Since both factors have the same parity, they must be even. In particular, the second factor is at least 2, and the first factor is most 648. This difference is achieved when $n=323$ and $y^2=289$, so the answer is $72+2(289)=650$.

11. There are N ways to decompose a regular 2019-gon into triangles (by drawing diagonals between the vertices of the 2019-gon) such that each triangle shares at least one side with the 2019-gon. What is the largest integer a such that 2^a divides N?

Answer: 2014

Solution: Note that there are 2017 triangles in any triangulation. Of these, 2 share 2 sides each with the 2019-gon, and the other 2015 share only 1 side each with the 2019-gon. There are 2019 ways to choose the first triangle with 2 sides on the 2019-gon. There is then 1 "interior" side of this triangle. There are 2 ways to make the triangle that uses this side. Continue this process until the 2019-gon is completely triangulated. Since we made 2015 "choices," there are $2019\cdot 2^{2015}$ ways to do this. Note that we double-counted (we we could have started with either of the 2 special triangles), so we divide by 2 to get $N=2019\cdot 2^{2014}$.

12. Anna has a 5×5 grid of pennies. How many ways can she arrange them so that exactly two pennies show heads in each row and in each column?

Answer: 2040

Solution: We consider two cases. First, suppose there exist a pair of rows and a pair of columns such that all 4 "intersection" squares contain pennies. There are $\binom{5}{2}^2 = 100$ ways to choose the rows and columns. Furthermore, brute force (or a modification of the argument that follows) shows that there are 6 ways to place the remaining 6 coins in the intersection of the remaining 3 rows and 3 columns. So this first case gives 600 possibilities. Second, suppose there are no such pairs of rows and columns. There are $\binom{5}{2} = 10$ ways to choose the locations of the coins in the first row. Look at the coin that is farther to the right. Then choose a different row to contain the other row in that column. Then choose a different column to have the other coin in that row. Continue this process until we have gone through all of the coins. There are $4! = 24$ ways to choose the order of the rows that we visit, and there are $3! = 6$ ways to choose the order of the columns that we visit. So this case gives 1440 possibilities. Summing the contributions from the two cases gives a total of $600 + 1440 = 2040$.